Gravitational Waves Detection

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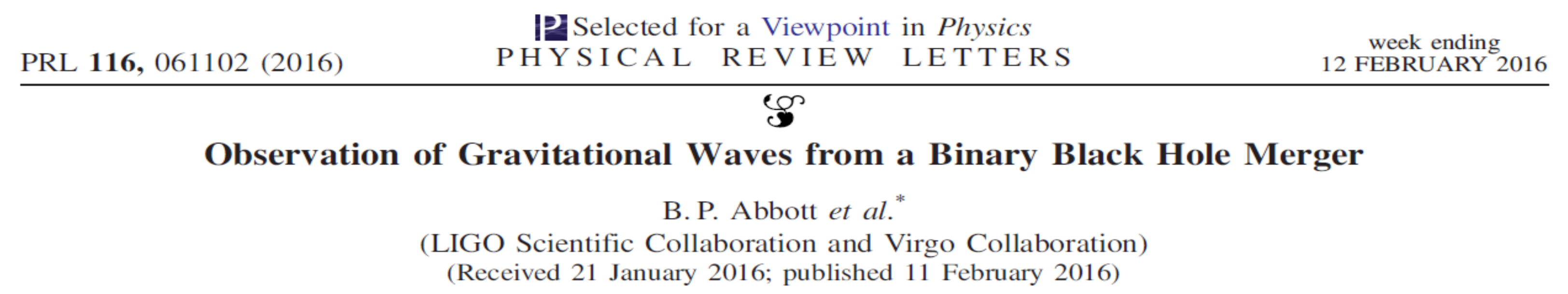
- Gravity as a fundamental force
- General Relativity
- Radiation of Gravitational Waves
- Indirect evidence (1974)
- Direct Detection (2016)

GW150914, GW151226, (LVT151012)

Disclaimer: the speaker is not one of the discoverers



PRL **116,** 061102 (2016)

Selected for a Viewpoint in *Physics*
PHYSICAL REVIEW LETTERS

week ending
12 FEBRUARY 2016

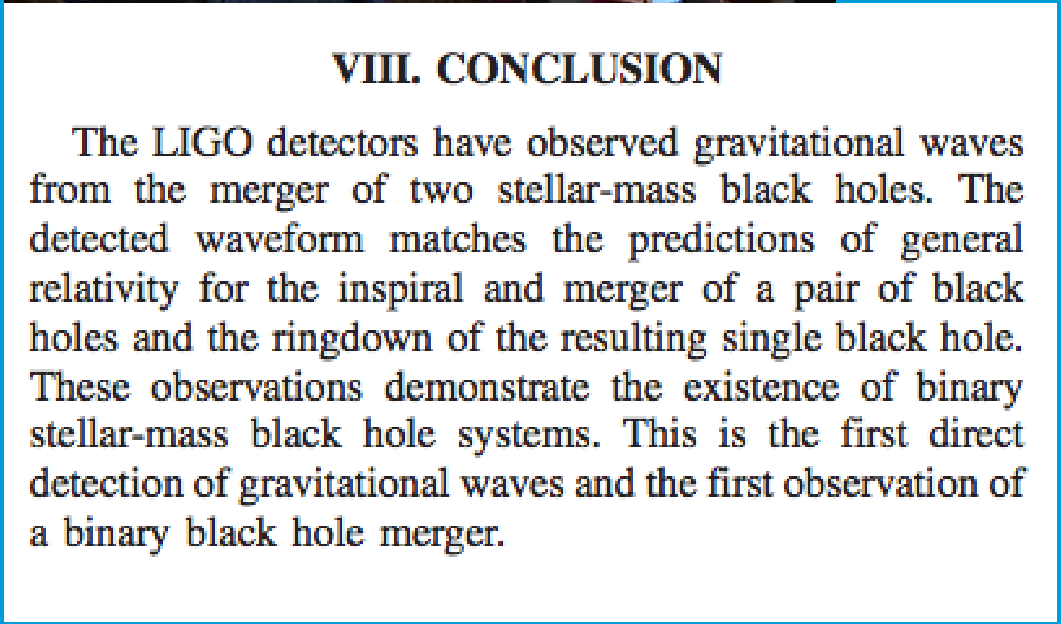
Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.**
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

11 Febbruary 2016 Press Conference

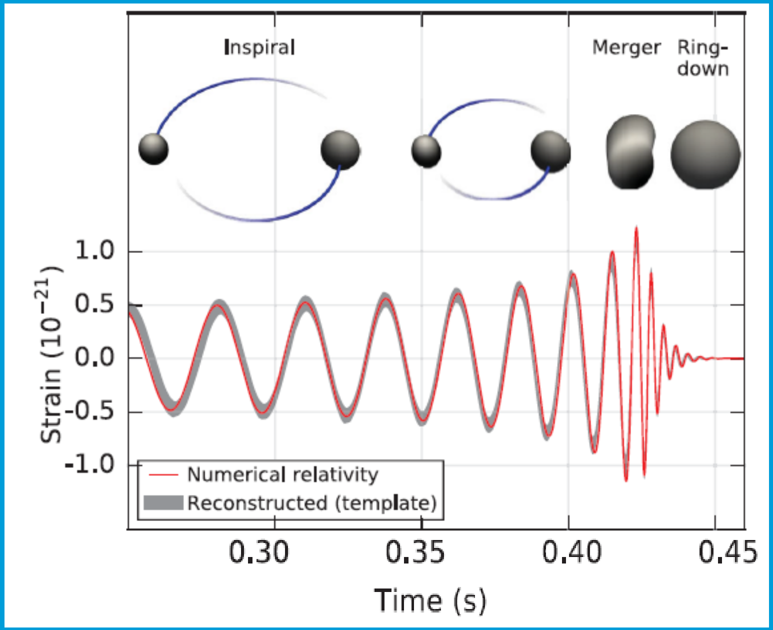
National Science Foundation

Washington DC - USA



VIII. CONCLUSION

The LIGO detectors have observed gravitational waves from the merger of two stellar-mass black holes. The detected waveform matches the predictions of general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.



1. Gravity as a fundamental force

Gravitational Interaction: classical theory (A. Einstein) in 1915. Responsible for the stability of matter on macroscopic scales.

- **Gravity as a fundamental force**
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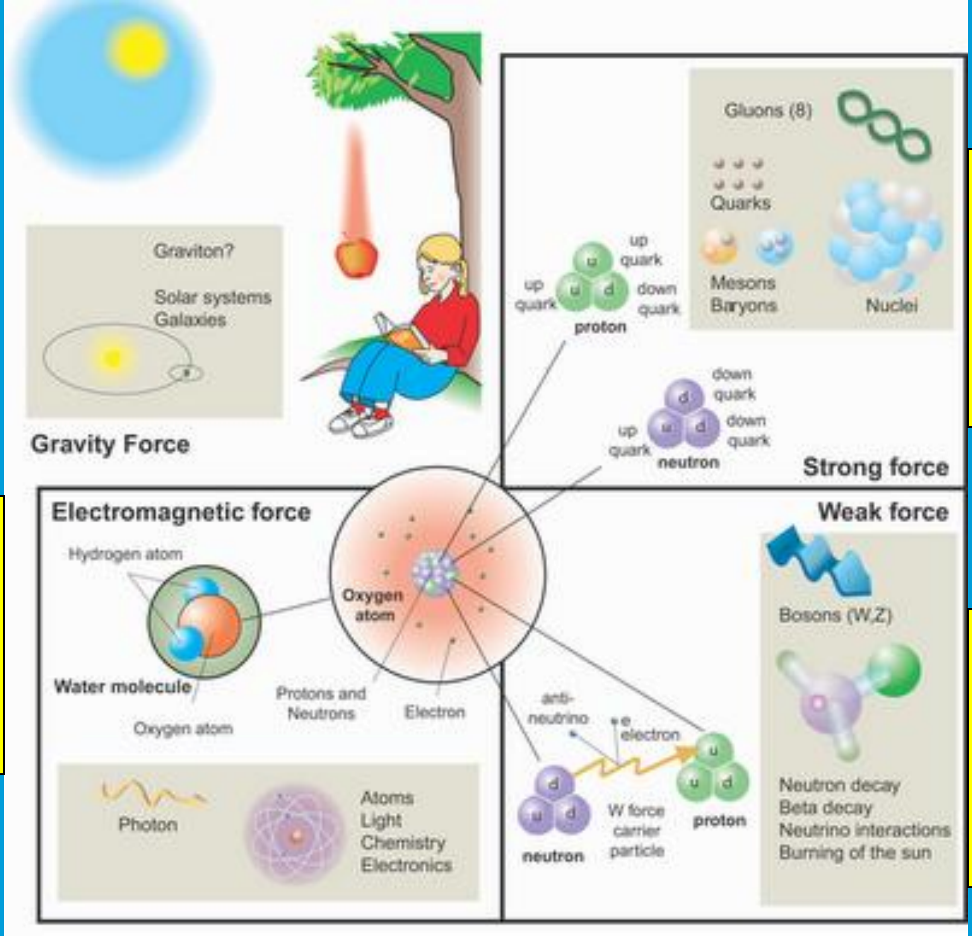


Strong Nuclear Interaction: a short range force, confined to a length of 10^{-15} m.

Elettromagnetic Interaction: takes place between charged particles (responsible for atomic stability).

Weak Nuclear Interaction: a subnuclear short-range (10^{-16} m) force.

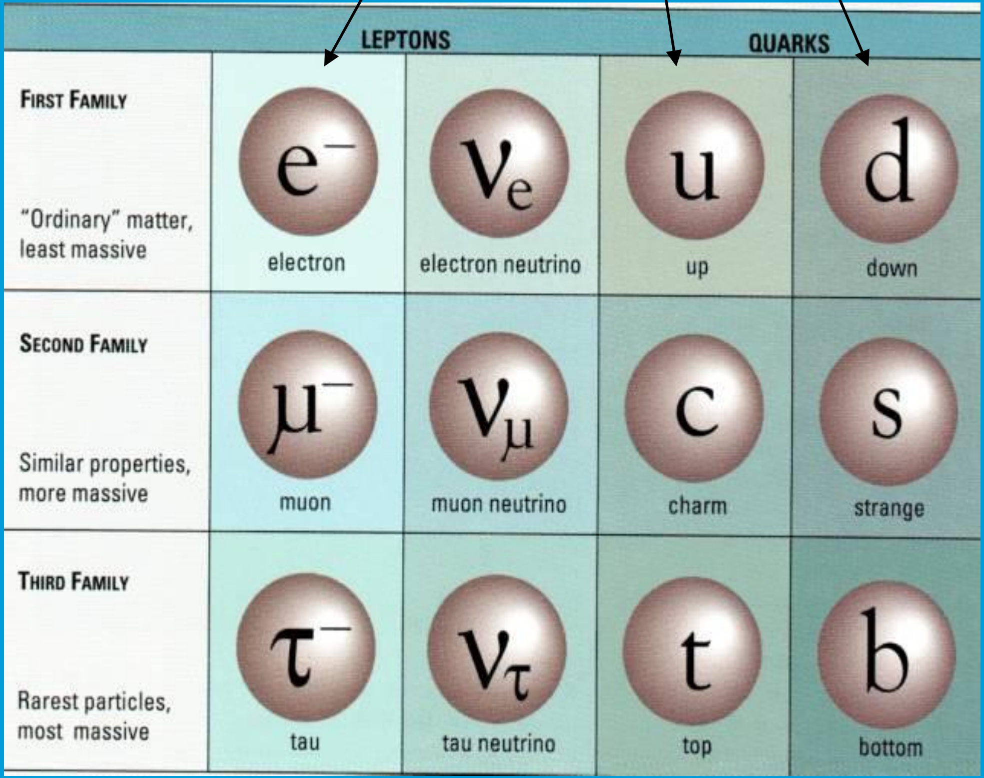
Quark and Leptons: the fundamental matter constituents

Elementary (structureless) down to 10^{-18} m or more

Well defined charge and spin

Ordinary Matter

Matter in ordinary (Earth-like) energy conditions



	LEPTONS		QUARKS	
First Family "Ordinary" matter, least massive	e^- electron	ν_e electron neutrino	u up	d down
Second Family Similar properties, more massive	μ^- muon	ν_μ muon neutrino	c charm	s strange
Third Family Rarest particles, most massive	τ^- tau	ν_τ tau neutrino	t top	b bottom

M
a
s
s

Make up unstable particles

Decay to stable particles

These are the consituents. How do they interact between them?

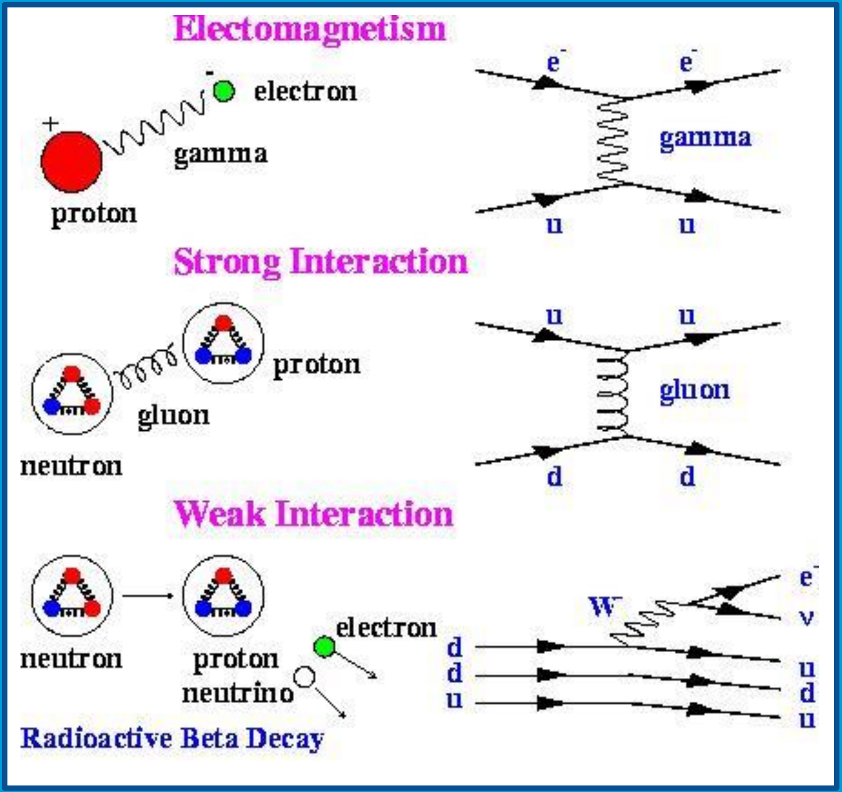
The concept of elementary interaction

Newton — Action at a distance

$$F = \frac{k}{r^2}$$

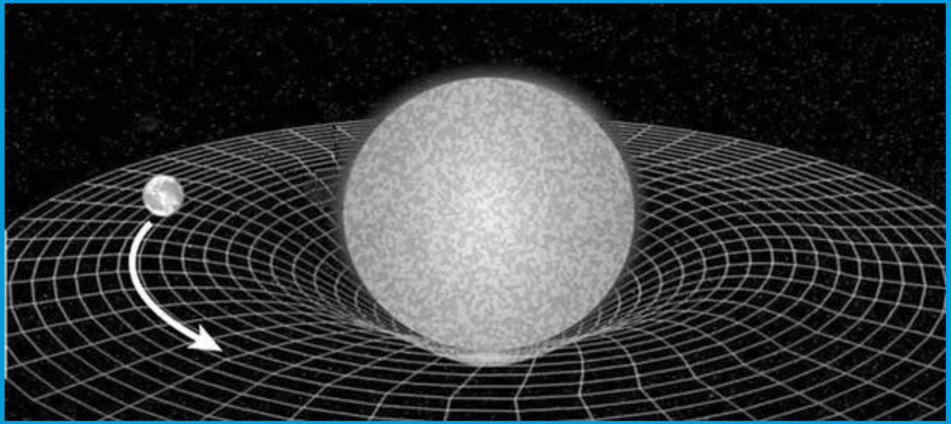
Faraday,Maxwell — The classical field

Quantum Fields (quanta exchange)



The only non-quantum force

Gravity (spacetime curvature)



Gravity is a special case :

- Gravity is way less intense than other forces on microscopic scales
- Gravity behaves the same for all bodies
- There is no Quantum Theory of Gravity
- Gravity is the decisive dynamical force in the Universe

$$G = \frac{2.12\times10^{15}}{kg^2}\hbar c$$

$$\alpha = \frac{e^2}{\hbar c} = \frac{(4.8\times10^{-10})^2}{1.054\times10^{-27}\times3\times10^{10}} \frac{dyne\,cm\,cm}{erg\ s\frac{cm}{s}} = \frac{1}{137}$$

Comparing gravity and electromagnetism:

$$\frac{2.12\times10^{15}}{kg^2}\frac{m\,m}{r^2}\hbar c \quad ?? \quad \frac{\alpha}{r^2}\hbar c$$

$$\frac{2.12\times10^{15}}{kg^2}mm \quad ?? \quad \alpha$$

Need to choose charges and masses. For the case of two protons

$$\frac{2.12\times10^{15}}{kg^2}(1.67\times10^{-27}kg)^2 \quad ?? \quad \alpha$$

$$5.9\times10^{-39} << \frac{1}{137}$$

….gravity weaker by many orders of magnitude

2. General Relativity

- Gravity as a fundamental force
- **General Relativity**
- Radiation of Gravitational Waves
- Indirect evidence (1974)
- Direct Detection (2016)

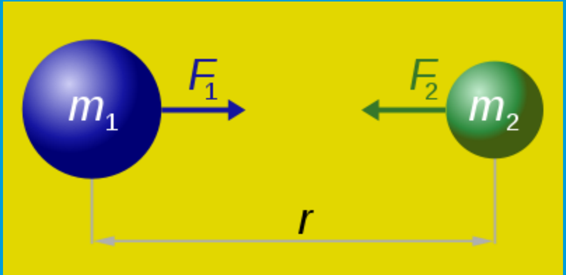
Gravity concerns all forms of energy of the Universe (mass included)
Responsible of bounds between macroscopic bodies

Classical field theory (Newton, 1687) for the case of masses

$$\nabla^2 \phi = 4\pi G \rho$$

Gravitational potential

Mass density



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

“Geometrized” spacetime field theory (Einstein, 1915)

General Relativity

Principle of Equivalence between inertial mass (inertia to force) and gravitational mass (gravitational charge) → gravity as a property of the spacetime background

Far away from sources of mass/energy (in a flat spacetime)

$$g_{\alpha\beta}(x) \rightarrow \eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Einstein Tensor

Energy-Momentum Tensor

Metric Tensor

$$G_{mn} = \frac{8pG}{c^4} T_{mn}$$

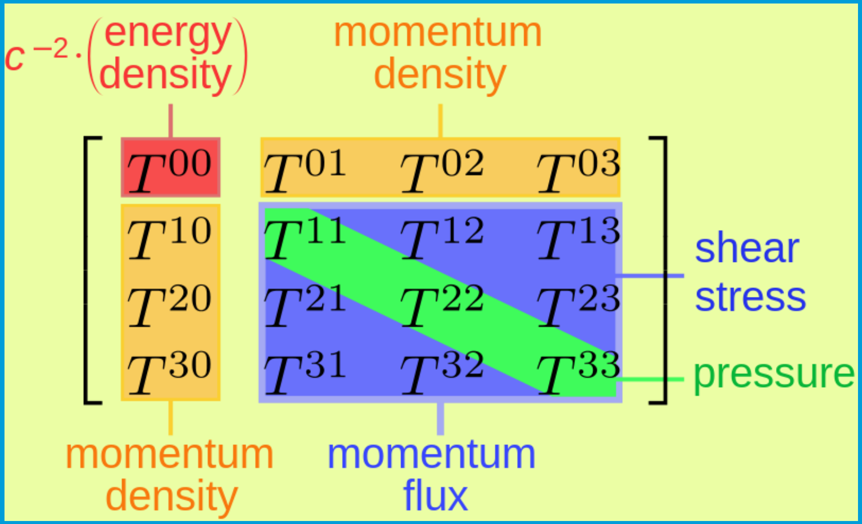
$$G_{\mu\nu} = G_{\mu\nu}(\partial_\theta \partial_\varepsilon g_{\alpha\beta})$$

Einstein’s Field Equations

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$$

Ten coupled differential equations

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$$



$$\begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$$

$$R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu} \right)$$

Determine the evolution of g based on initial conditions. They are not independent equations because of the Bianchi Identity

$$G^{\alpha\beta}{}_{;\beta} = 0$$

Six really independent equations to determine the six independent functions (among the ten g components) that characterize the geometry independent of coordinates

3. Radiation of Gravitational Waves

- Gravity as a fundamental force
- General Relativity
- **Radiation of Gravitational Waves**
- Indirect evidence (1974)
- Direct Detection (2016)

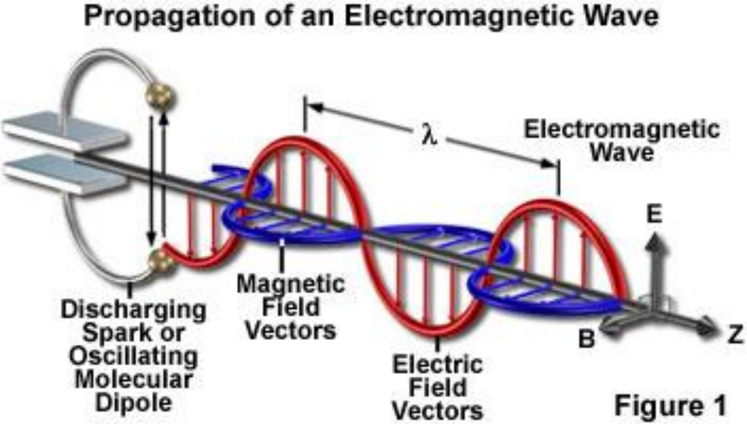
Electromagnetic radiator

- Oscillating Dipole
- E.M. waves discovered in 1886 (Hertz).

$$\left(\nabla^2 - \partial_t^2\right)A^\mu = 0$$

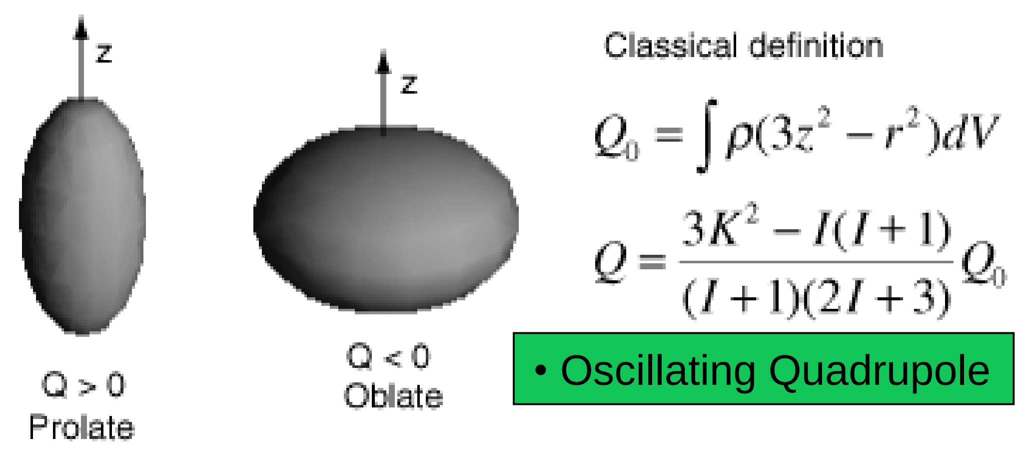
4-vector

- Two polarization states
- Photon: spin 1



Gravitational radiator

Dipole gravitational radiation in GR is not present because of conservation of the linear momentum of N-body system

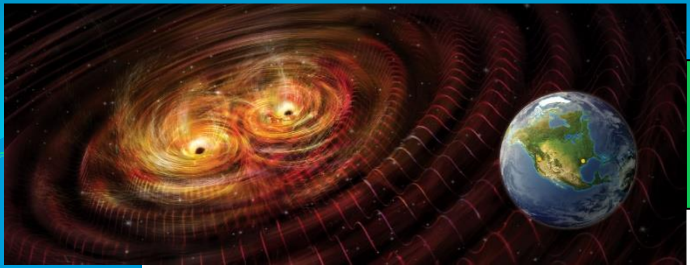


Classical definition

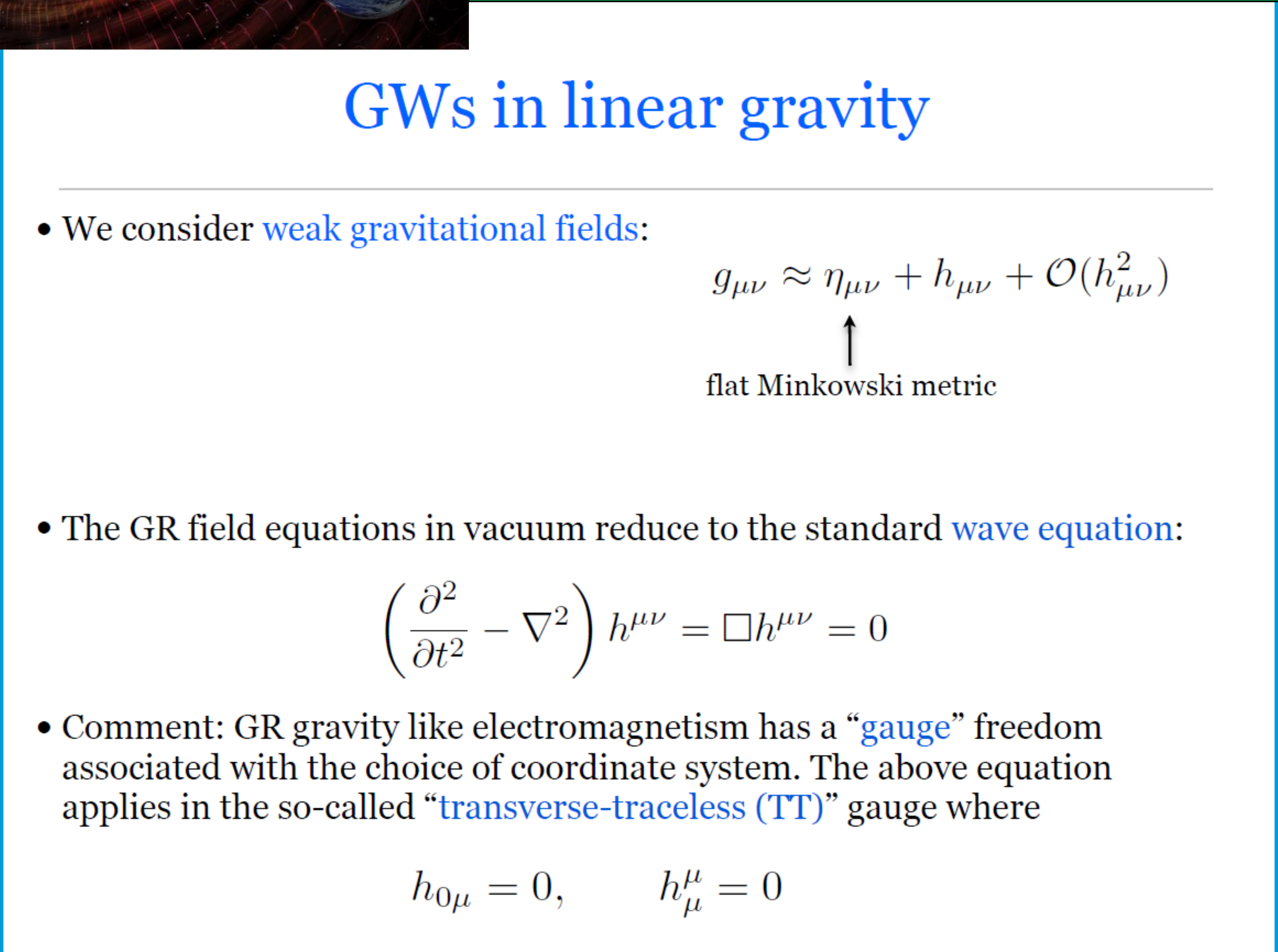
$$Q_0 = \int \rho(3z^2 - r^2)dV$$

$$Q = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0$$

- Oscillating Quadrupole



- A Strong faraway Source (not treated in detail)
- A little perturbation (at the Earth)



GWs in linear gravity

- We consider weak gravitational fields:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2_{\mu\nu})$$

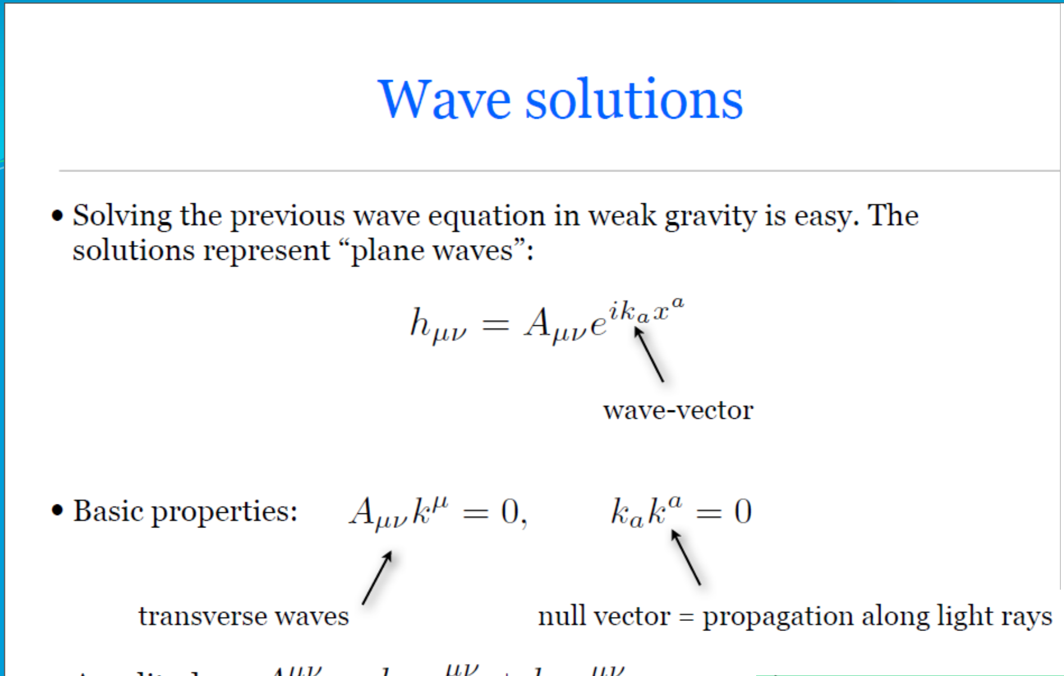
flat Minkowski metric

- The GR field equations in vacuum reduce to the standard wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) h^{\mu\nu} = \Box h^{\mu\nu} = 0$$

- Comment: GR gravity like electromagnetism has a "gauge" freedom associated with the choice of coordinate system. The above equation applies in the so-called "transverse-traceless (TT)" gauge where

$$h_{0\mu} = 0, \qquad h^{\mu}_{\mu} = 0$$



Wave solutions

- Solving the previous wave equation in weak gravity is easy. The solutions represent “plane waves”:

$$h_{\mu\nu} = A_{\mu\nu} e^{i \overset{\nwarrow}{k_a} x^a}$$

wave-vector

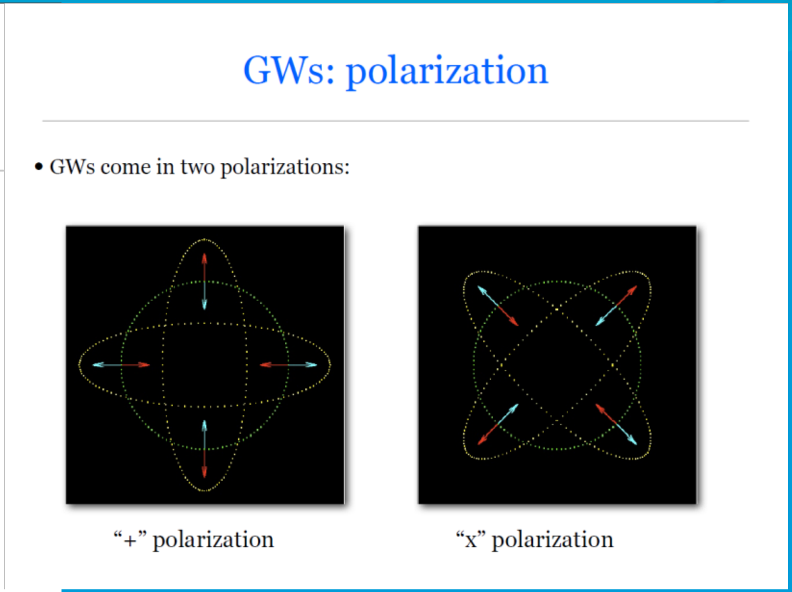
- Basic properties: $A_{\mu\nu} k^\mu = 0, \qquad k_a k^a = 0$

transverse waves

null vector = propagation along light rays

- Amplitude: $A^{\mu\nu} = h_+ e_+^{\mu\nu} + h_\times e_\times^{\mu\nu}$

two polarizations

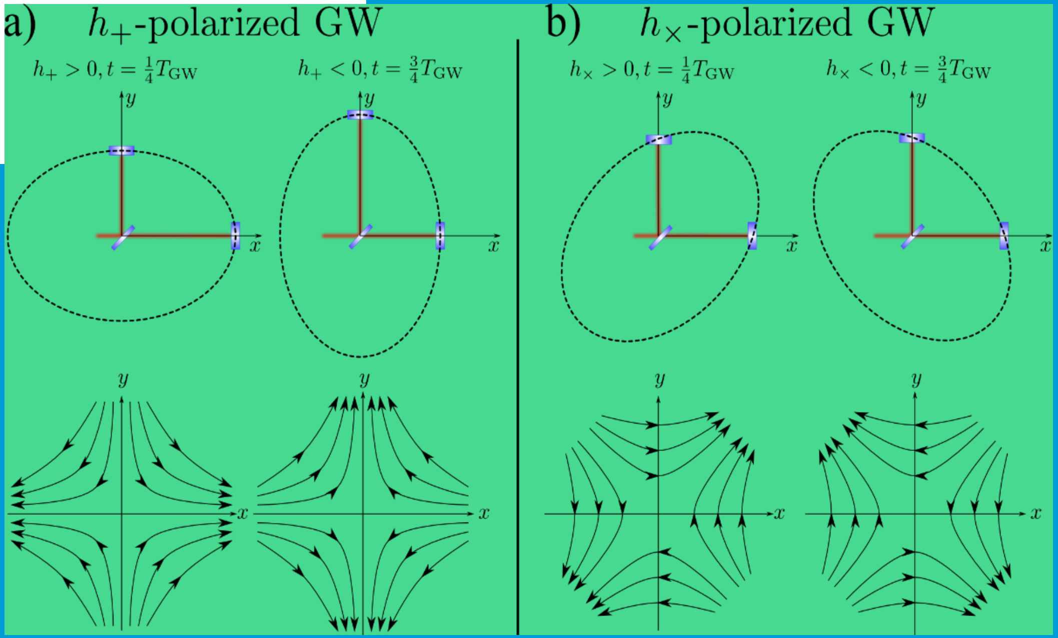


GWs: polarization

- GWs come in two polarizations:

“+” polarization

“x” polarization

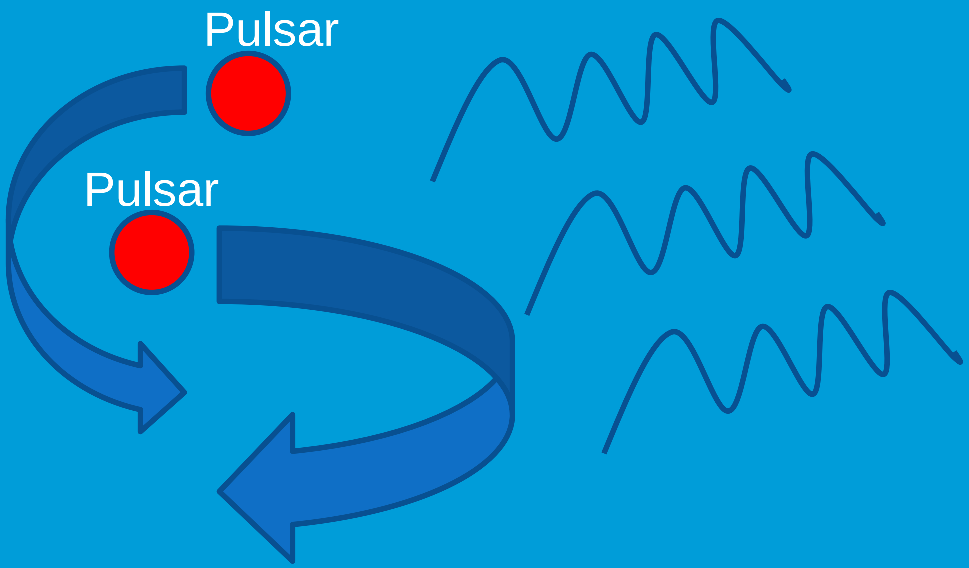


A Gravitational Wave entering the slide from above will alternatively deform the elliptical shape (where mirrors can be located)

4. Indirect evidence

- Gravity as a fundamental force
- General Relativity
- Radiation of Gravitational Waves
- **Indirect evidence (1974)**
- Direct Detection (2016)

Two massive and compact objects orbiting one around the other



The system can emit gravitational waves (if its quadrupole moment changes)

Energy conservation tells that the energy of the system decreases! The orbit parameters change

The effect can be detectable!

INDIRECT evidence of the existence of Gravitational Radiation from the source

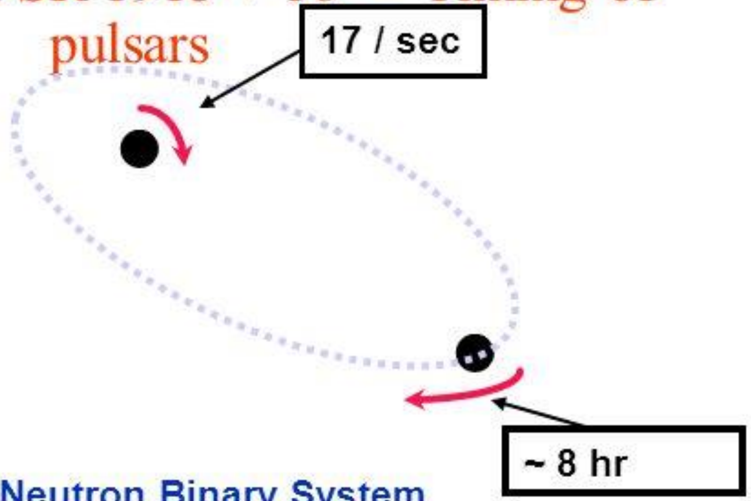
Delay in the
Time of Periastron
Of Binary Pulsar

Gravitational Waves

the evidence

Neutron Binary System – Hulse & Taylor

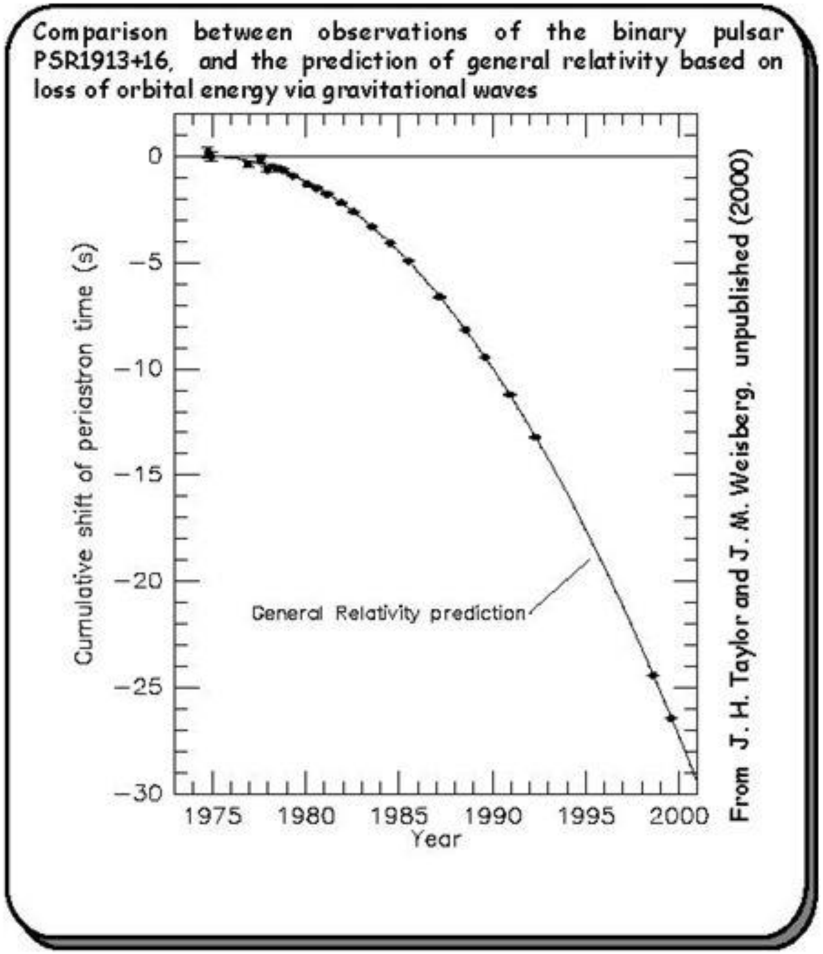
PSR 1913 + 16 -- Timing of pulsars



Neutron Binary System
- separated by 10^6 miles
- $m_1 = 1.4m_{\odot}$; $m_2 = 1.36m_{\odot}$; $\varepsilon = 0.617$

Prediction from general relativity
- spiral in by 3 mm/orbit
- rate of change orbital period

Emission of gravitational waves


Comparison between observations of the binary pulsar PSR1913+16, and the prediction of general relativity based on loss of orbital energy via gravitational waves

5. Direct Detection

Two massive and compact objects orbiting one around the other

- Gravity as a fundamental force
- General Relativity
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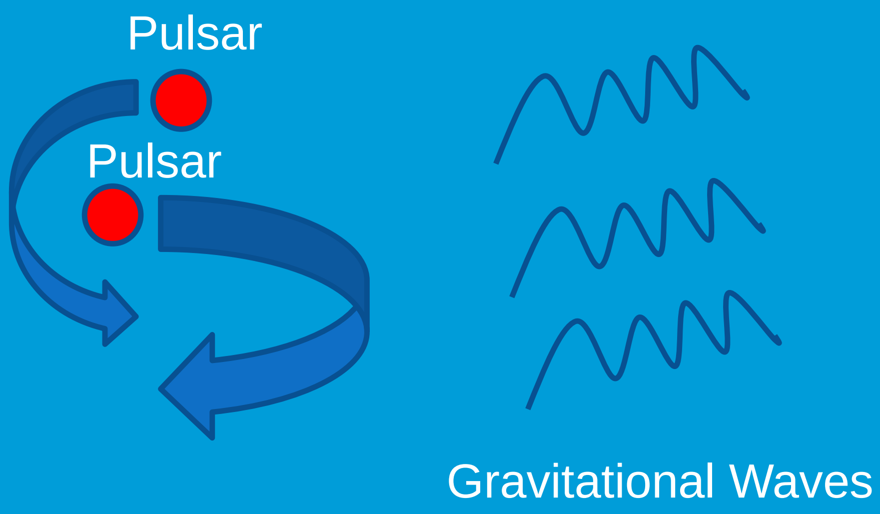
- **Direct Detection (2016)**



Detector on Earth

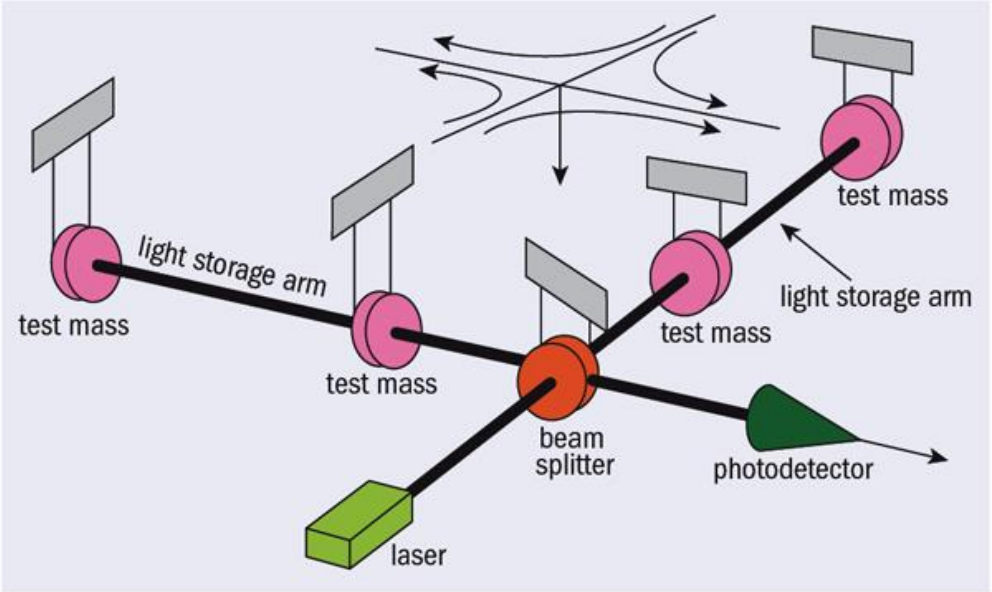
1960: J. Weber proposal of resonant detectors (cryogenic bars, 1960).

In Europe three of them have been built by E. Amaldi and coworkers (CERN, Legnaro and Frascati).

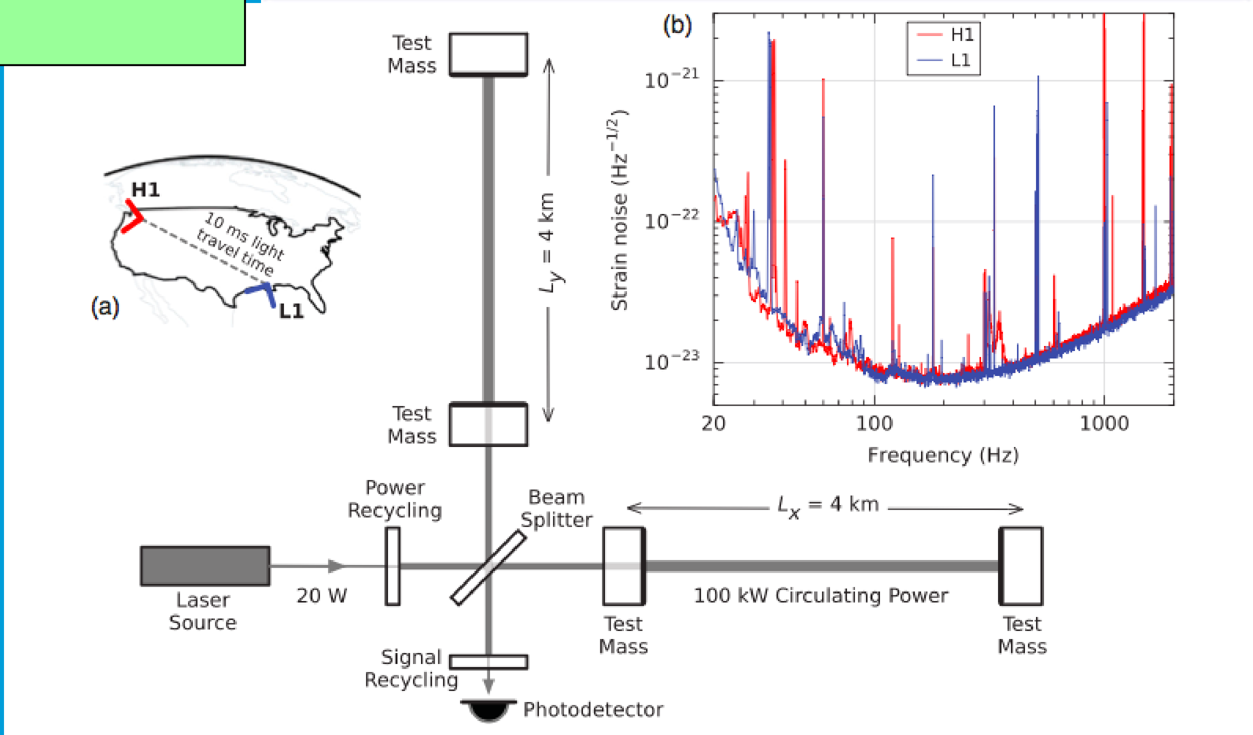
The search for Gravitational Waves: Michelson Interferometers in the Fabry-Pérot configuration

Key idea: mirrors (test masses) whose position is monitored by laser

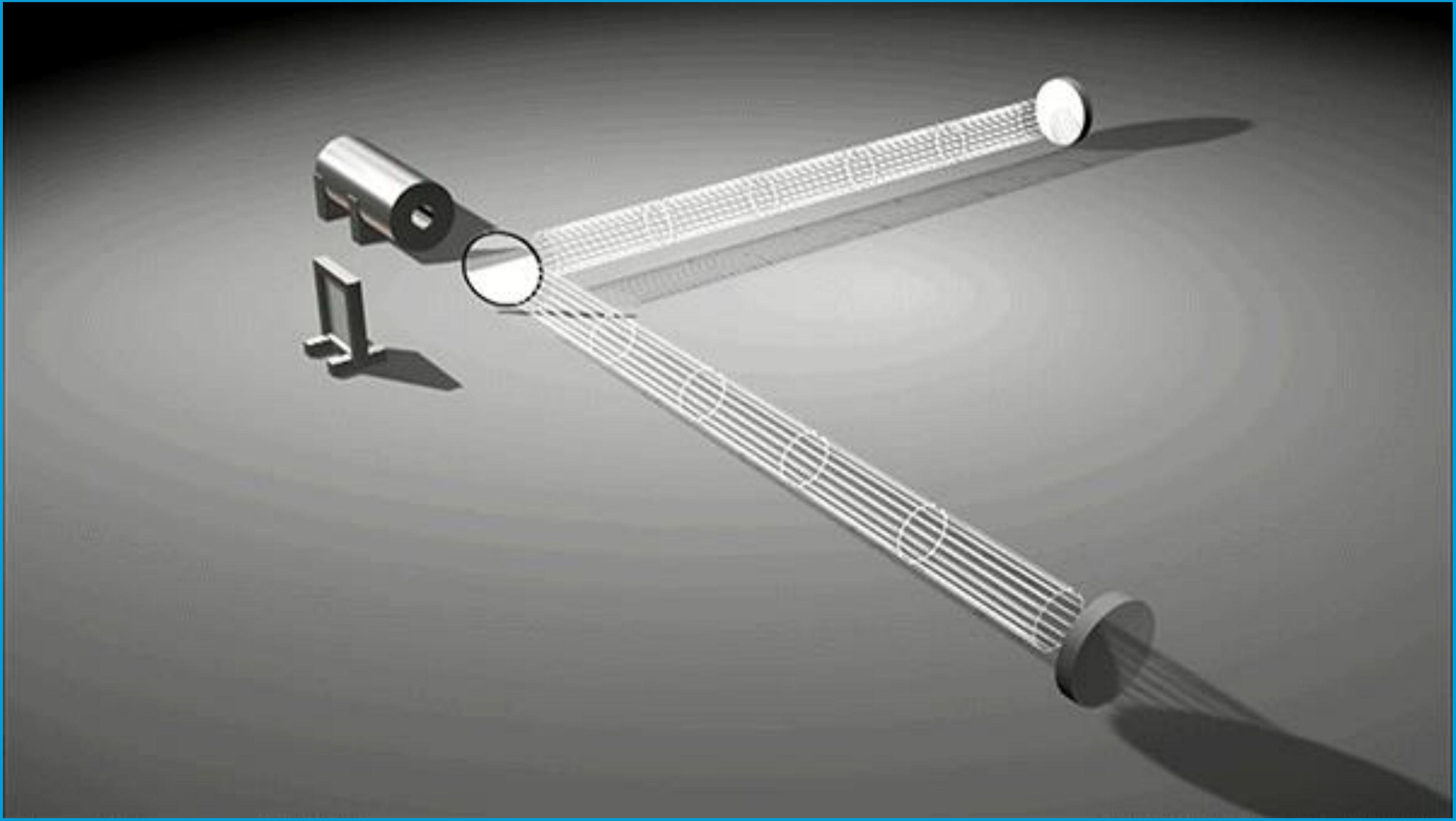
LIGO (USA, 2 interferometers) and Virgo (Italy-France) in a single 1000 scientist collaboration



Strain Sensitivity and location of Hanford and Livingston Interferometers (LIGO)



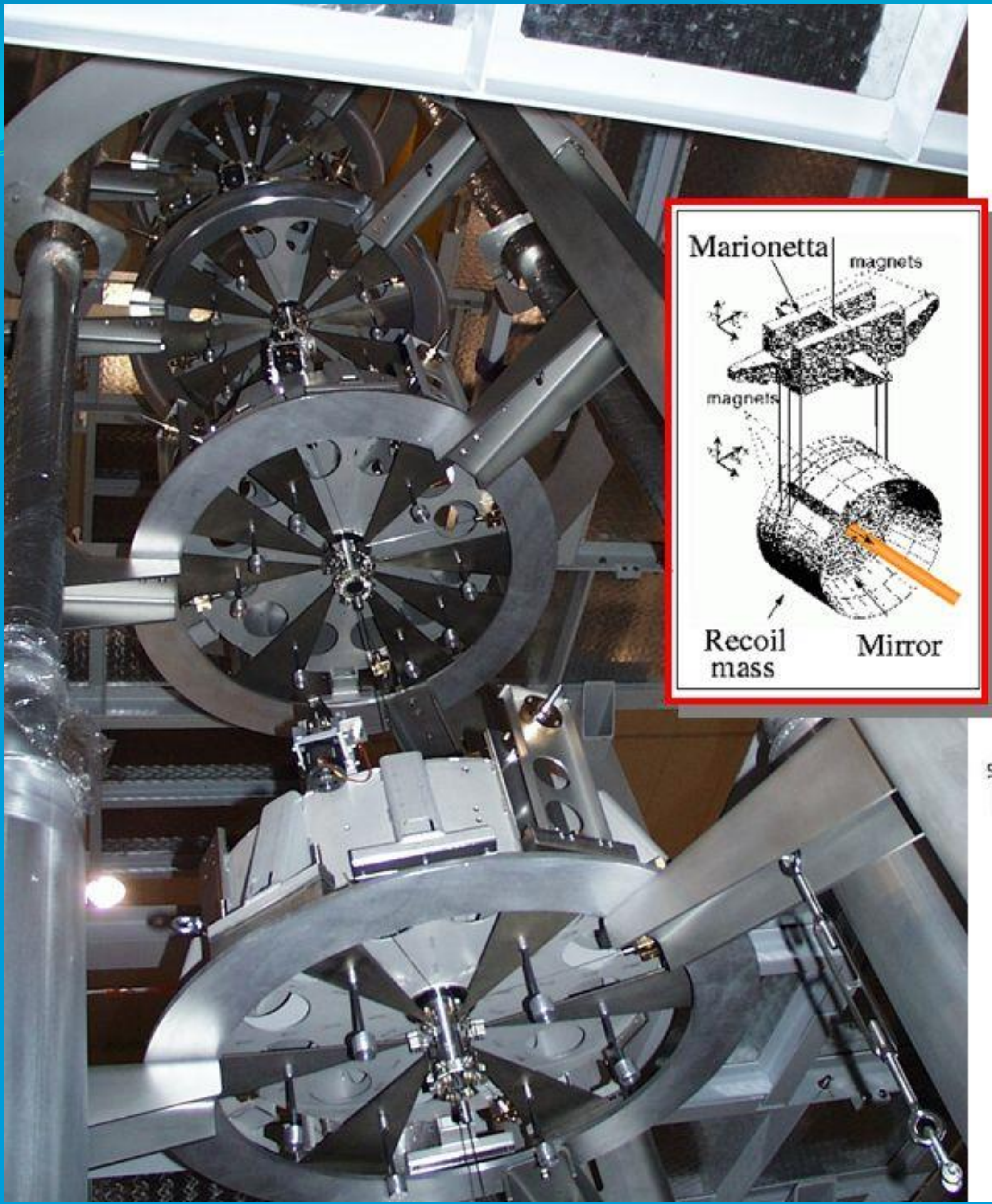
The search for Gravitational Waves: Michelson Interferometer to detect the «mirror position during the passage of the gravitational wave».



A gravitational wave will actually «create and destroy» space.

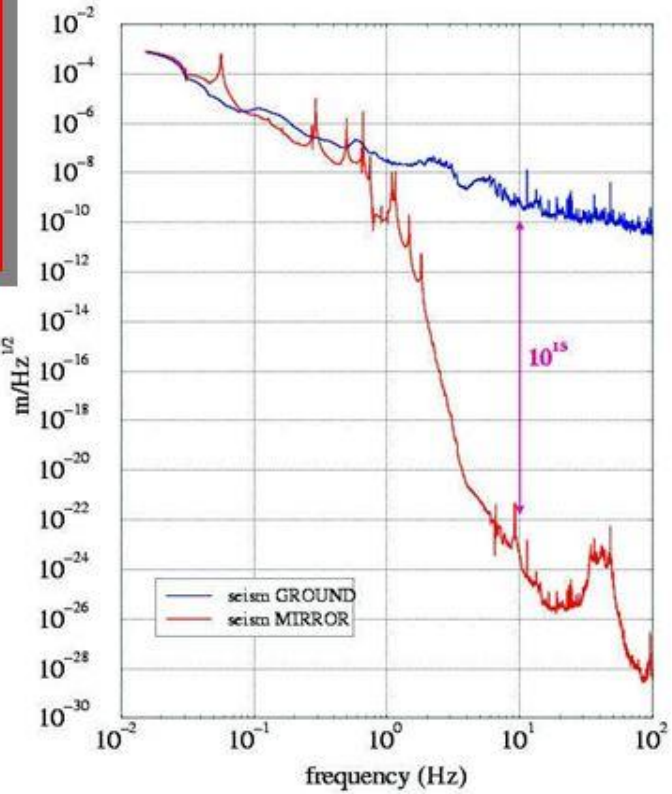
Inside one of the LIGO arms

Superattenuators



Possible contributions:

- Virgo+ will use monolythic suspension
- Input-mode cleaner suspension



14 Settember 2016: Hanford and Livingston observe at the same time (within 10 ms) a clear gravitational wave signal: GW150914, with full duration of 0.5 s.

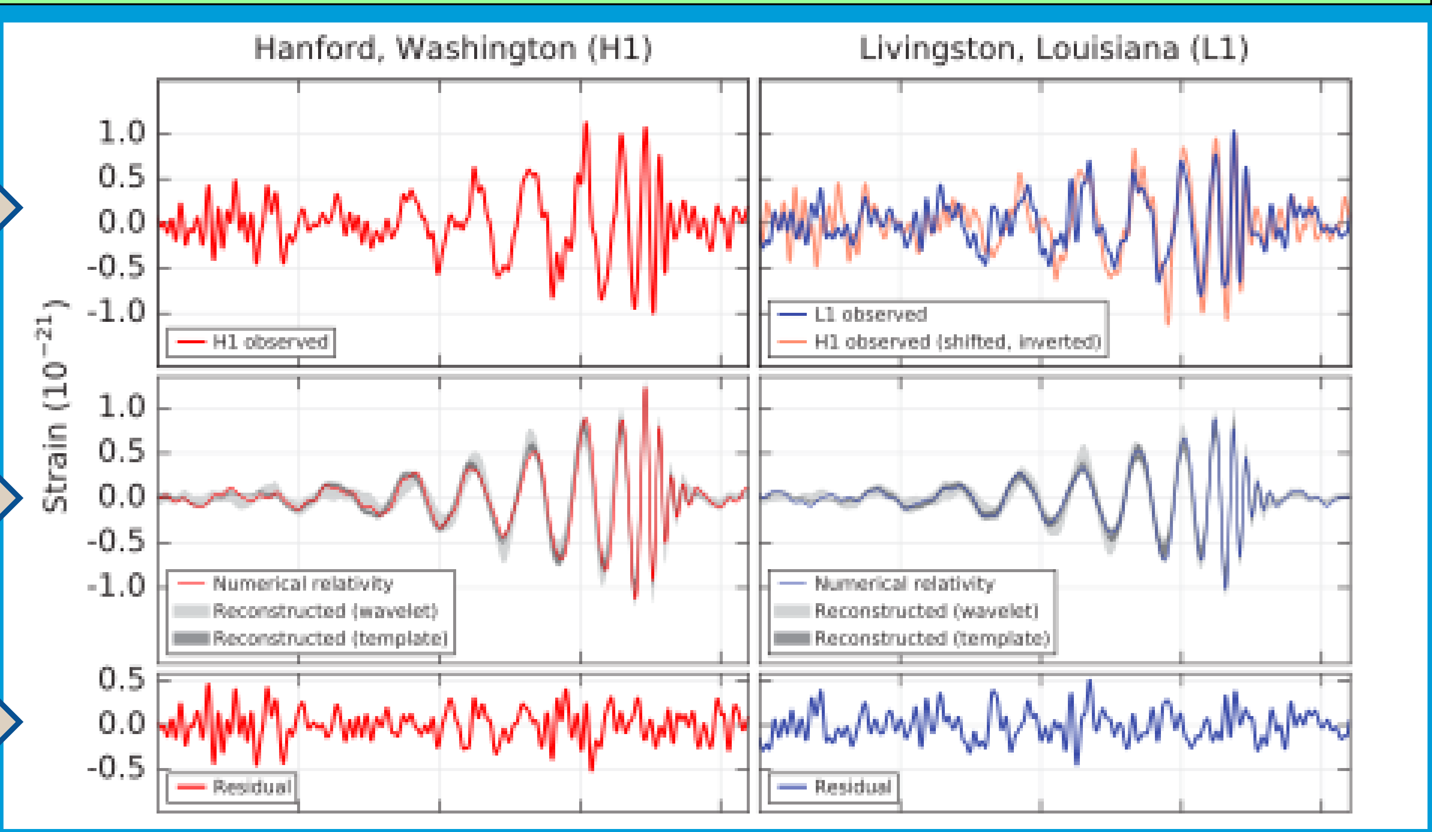
This kind of signal can be generated only by the mutual collapse of two Black-Holes having ~36 and 29 Solar Masses. The resulting (Kerr-type) Black Hole has 62 solar masses. Three solar masses have been transformed to energy (spacetime waves).

Physical Review Letters 116 (2016) 061102.

Raw Signals

Background subtracted

Background



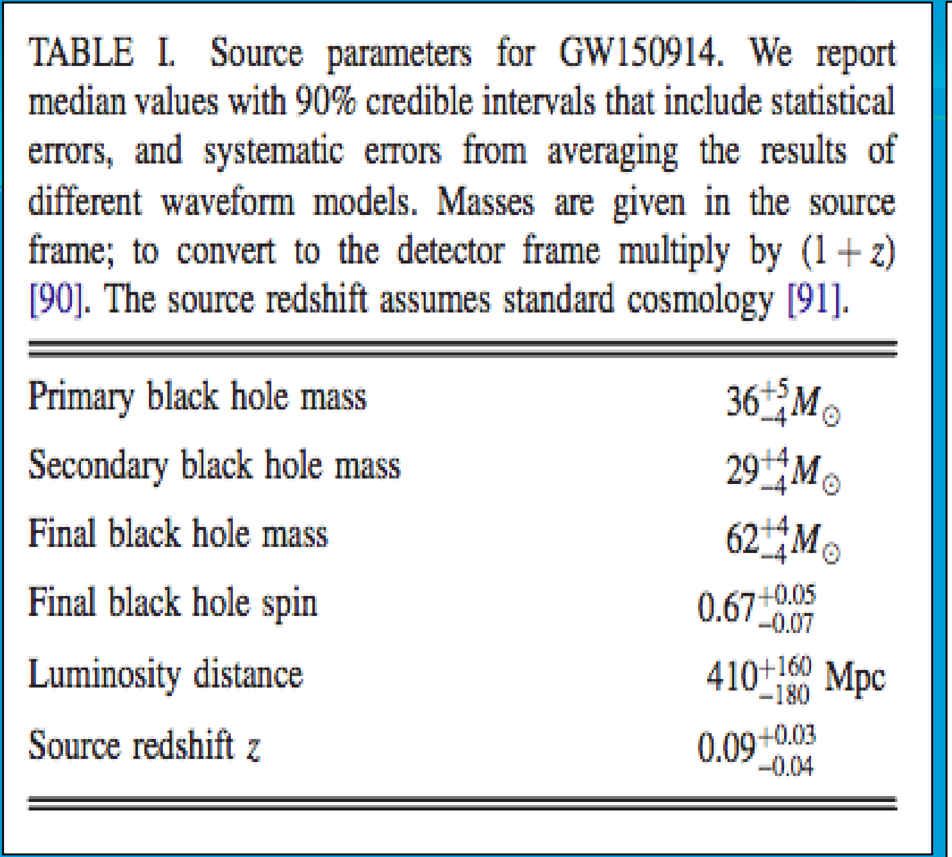
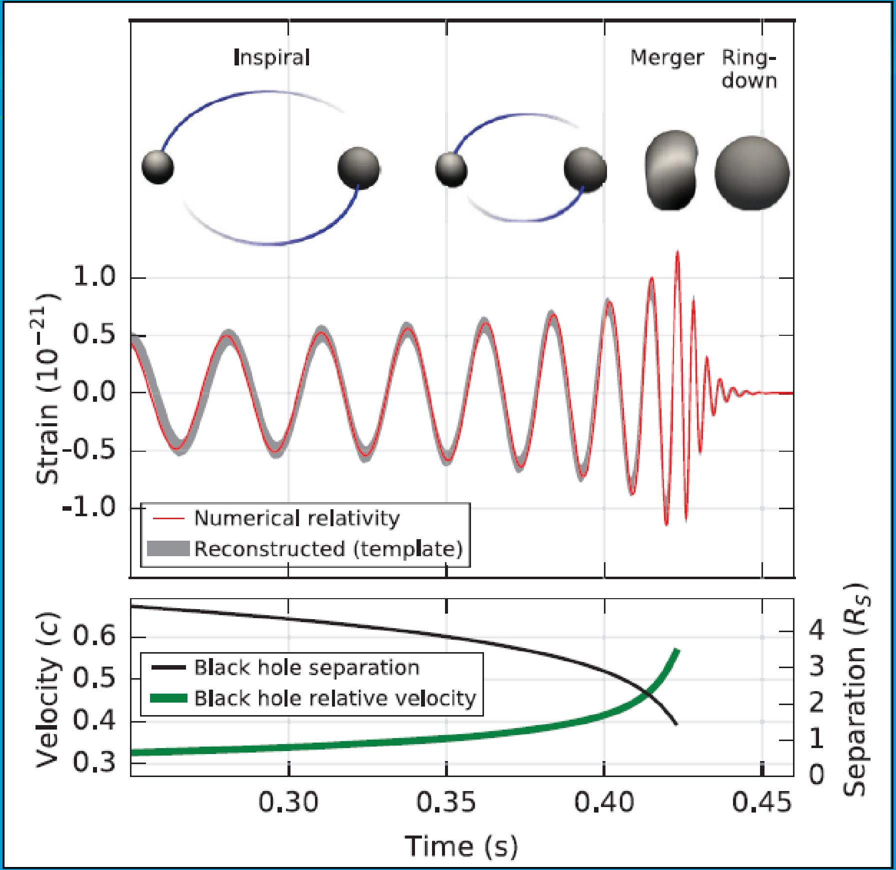


TABLE I. Source parameters for GW150914. We report median values with 90% credible intervals that include statistical errors, and systematic errors from averaging the results of different waveform models. Masses are given in the source frame; to convert to the detector frame multiply by $(1 + z)$ [90]. The source redshift assumes standard cosmology [91].

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$



The evidence in favor of a BH-BH scenario is compelling (only two BH's can reach 75 Hz of mutual rotational frequency before merging in a region of R_S » $200 \ km$

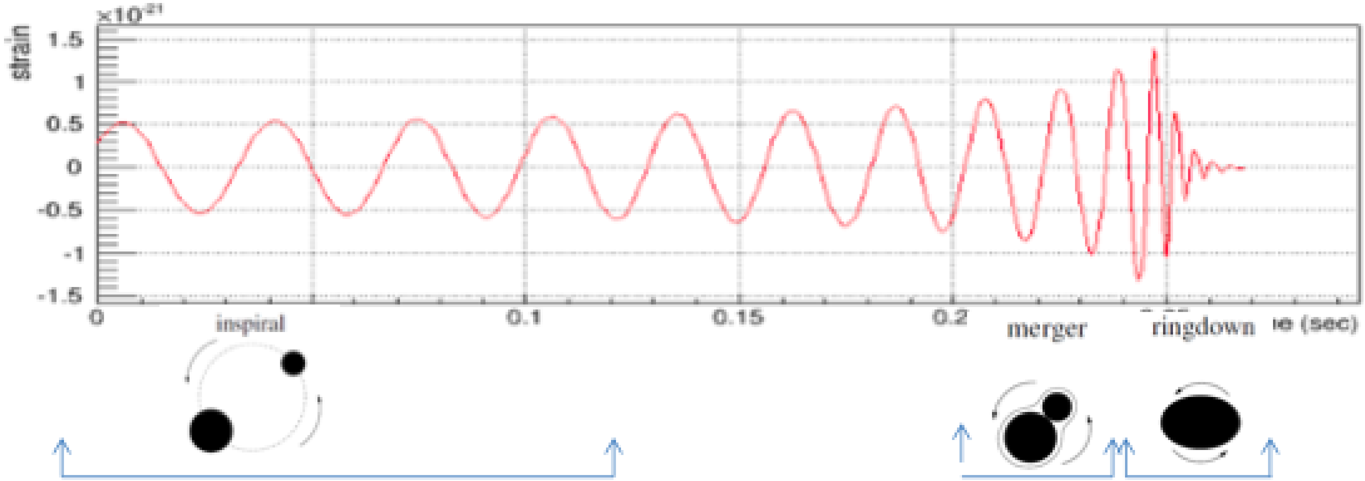
Maximum power transformed into gravitational waves is about 1000 times a Supernova energy release! $P_{\max} \approx 4 \times 10^{56} \ erg / s$

Spectacular confirmation of General Relativity as a classical theory.

Well predicted discovery (this slide from a 2014 Conference)!

Sources of gravitational waves

- ❑ Compact binary (BNS, BBH) coalescence. Best candidate for ground based detectors



The GW signal produced in the last few inspiraling cycles are expected to fall in the interferometer bandwidth

GW emission well approximated by the quadrupole formula. Analytical solution available. GW Candle.

Only numerical solution available

Perurbative or numerical

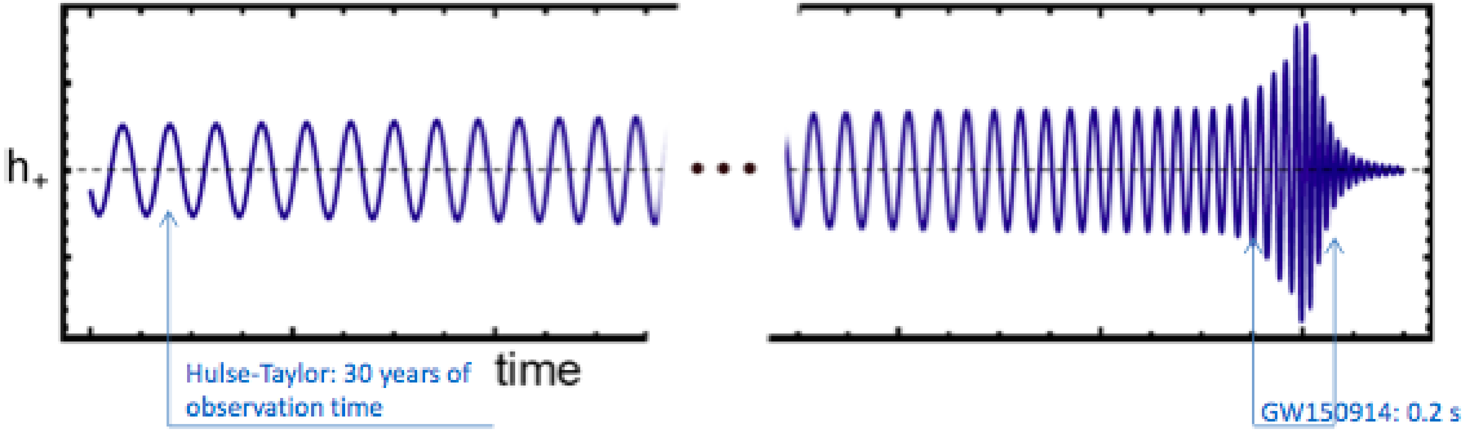
Test of general relativity in strong non-linear regime

Numerical (not Exact) General Relativity Solutions

How does GW150914 compare with the Hulse-Taylor binary pulsar?

PSR1916+13 versus GW150914

Waveform

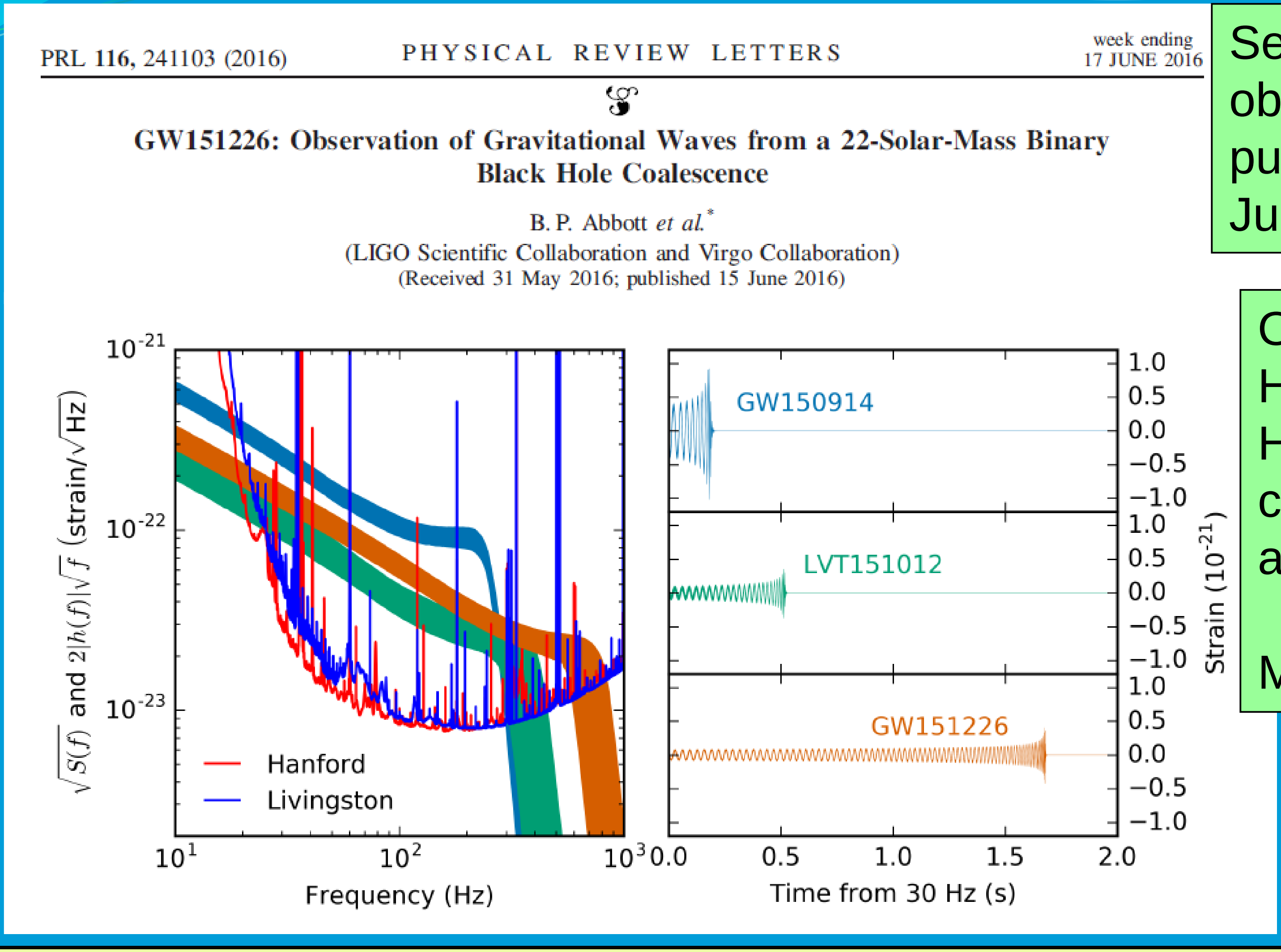


Hulse-Taylor: 30 years of observation time

GW150914: 0.2 s

PSR1916+13	**Binary system**	GW150914
NS-NS	**Compact object**	BH-BH
$M_1 = 1.44\,\mathrm{M}_\odot, M_2 = 1.3\,\mathrm{M}_\odot$	**Mass**	$M_1 = 36\,\mathrm{M}_\odot, M_2 = 29\,\mathrm{M}_\odot$
4×10^{-23}	**GW amplitude**	2×10^{-21}
7×10^{-5} Hz	**GW frequency**	$30 \div 300$ Hz
7.4×10^{9} years	**Time to merging**	$0.3 \div 0$ s
$6 \times 10^{30}\,\mathrm{erg\,s^{-1}}$	**Peak luminosity**	$3 \times 10^{56}\,\mathrm{erg\,s^{-1}}$
6.4 pc	**Distance**	410 Mpc
10^{6} km	**Radius orbit**	~ 200 km

Detection of other Gravitational Waves sources? Yes!



PRL 116, 241103 (2016) PHYSICAL REVIEW LETTERS week ending 17 JUNE 2016

GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence

B. P. Abbott *et al.**

(LIGO Scientific Collaboration and Virgo Collaboration)

(Received 31 May 2016; published 15 June 2016)

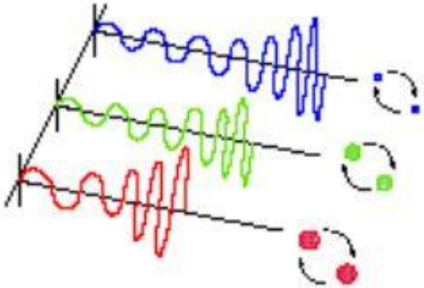
Second direct observation published in June 2016!

Observed Black Hole – Black Hole (BH-BH) coalescences are at least two.

Maybe three.

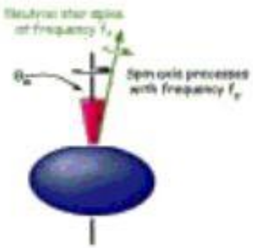
The expectation is for many sources to be identified in the near future.
A Gravitational Wave astronomy!

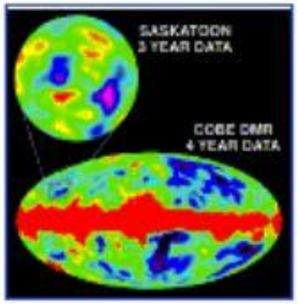
Astrophysical Sources for Terrestrial GW Detectors

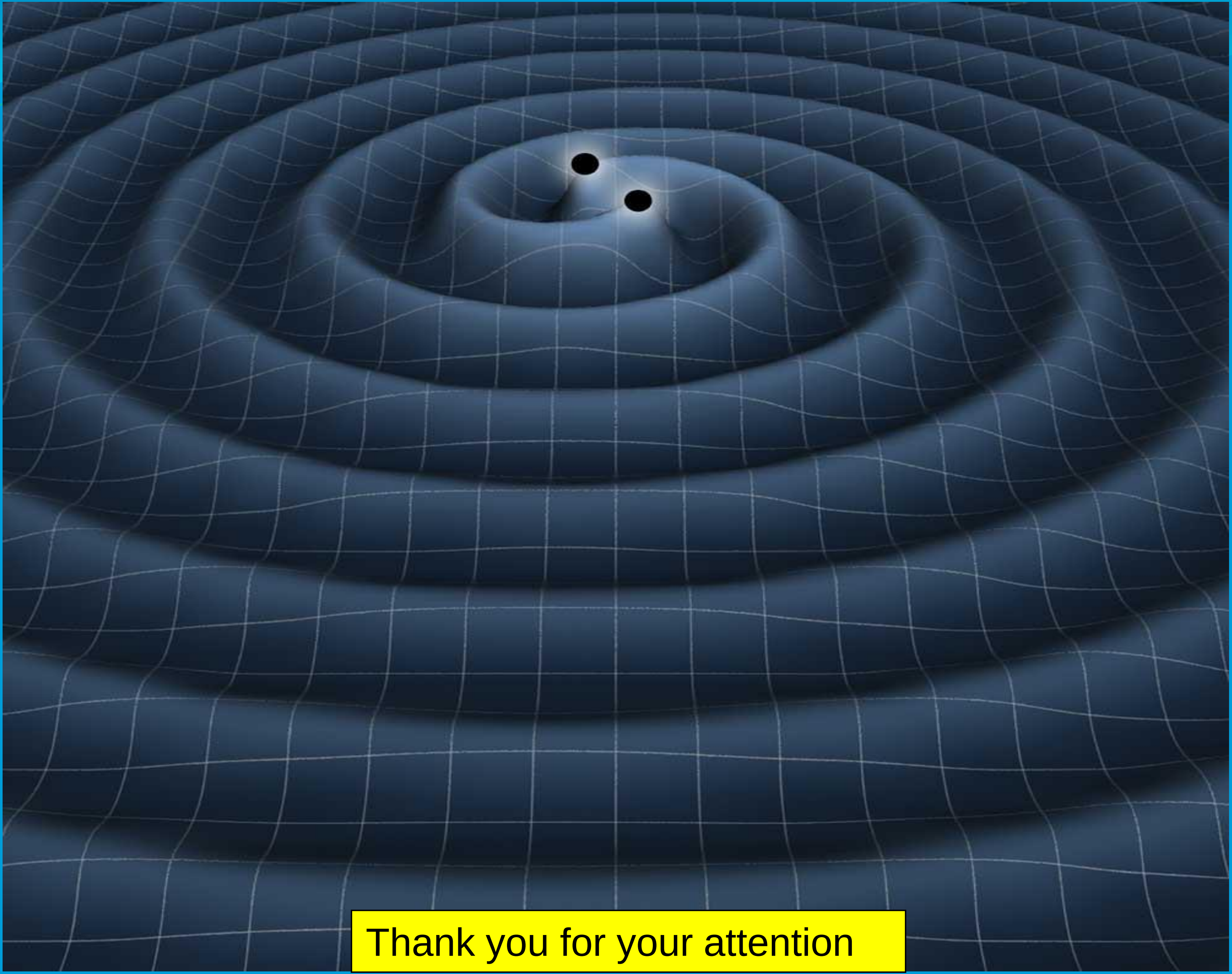


- Compact binary Coalescence: "chirps"
  - NS-NS, NS-BH, BH-BH
- Supernovas or GRBs: "bursts"
  - GW signals observed in coincidence with EM or neutrino detectors

- Pulsars in our galaxy: "periodic waves"
  - Rapidly rotating neutron stars
  - Modes of NS vibration
- Cosmological: "stochastic background" ?
  - Probe back to the Planck time (10^{-43} s)
  - Probe phase transitions : window to force unification
  - Cosmological distribution of Primordial black holes



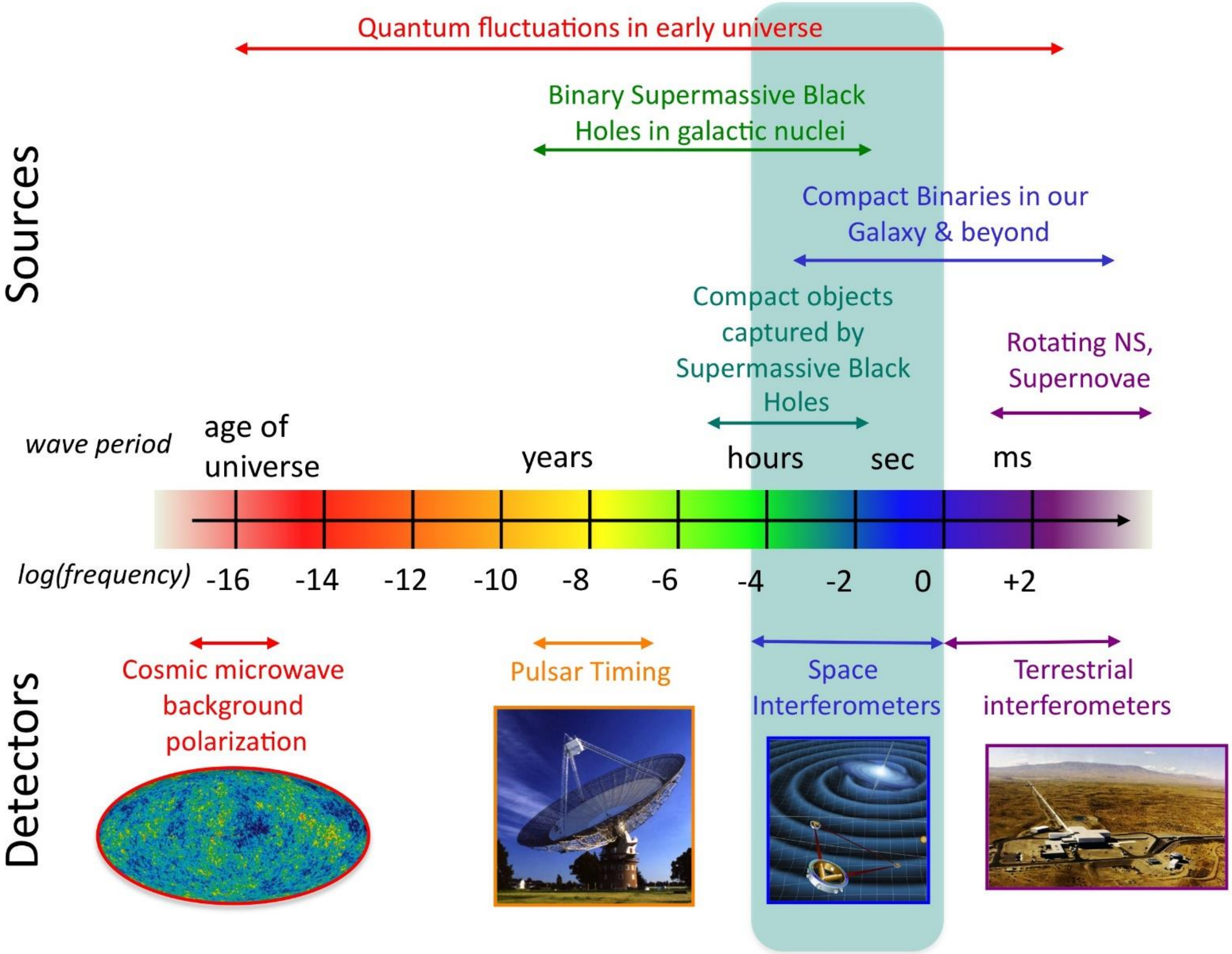




Thank you for your attention

Backup Slides

The Gravitational Wave Spectrum

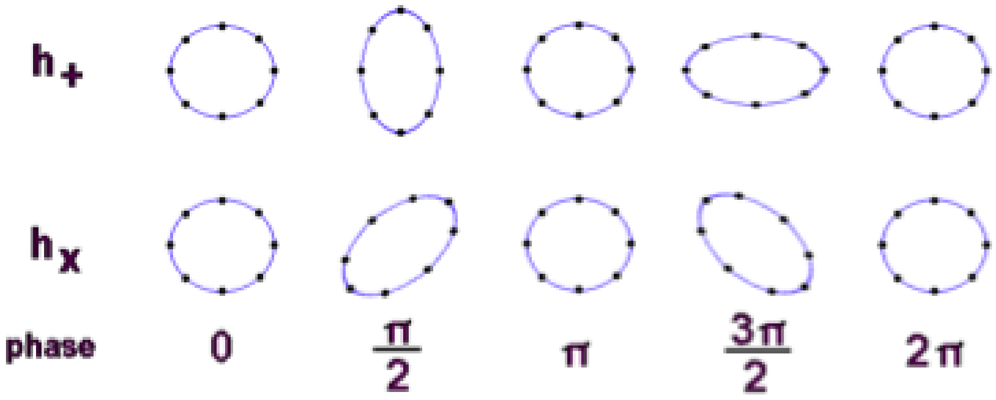


Gravitational Waves and G.R. (1916)

- Gravity is a manifestation of curvature of space-time produced by matter-energy
- Any rapidly moving mass generates fluctuations in spacetime curvature which propagate at the speed of light. **Gravitational waves**
- The physical quantity transported by gravitational waves is curvature

GW properties

- Speed of light,
- Transverse, traceless,
- GWs stress and compress spacetime in two directions,
- Two polarizations states "+" and "x".



$$\boxed{\frac{\Delta L}{L} \sim h}$$

Sources of gravitational waves

A mass distribution with $\dddot{Q}_{\mu\nu} \neq 0$ ($Q_{\mu\nu}$ mass-quadrupole) produces GWs

$$P = \overset{\text{Very low}}{\boxed{\frac{G}{5c^5}}} \dddot{Q}_{\mu\nu}\dddot{Q}^{\mu\nu}$$

Even in the most optimistic case $h \lesssim 10^{-20}$

☐ Burst (i.e. core collapse supernovae)

$$h \sim 6 \times 10^{-21} \left(\frac{E}{10^{-7}\,\mathrm{M}_\odot c^2}\right)^{\frac{1}{2}} \left(\frac{1\,\mathrm{ms}}{T}\right) \left(\frac{1\,\mathrm{kHz}}{f}\right) \left(\frac{10\,\mathrm{kpc}}{r}\right)$$

☐ Continous waves (i.e non axisymmetric spinning neutron star)

$$h_0 = \frac{4\pi^2 G I_{zz} f_{GW}^2}{c^4 r}\epsilon = (1.1 \times 10^{-24}) \left(\frac{I_{zz}}{I_0}\right) \left(\frac{f_{GW}}{1\,\mathrm{kHz}}\right)^2 \left(\frac{1\,\mathrm{kpc}}{r}\right) \left(\frac{\epsilon}{10^{-6}}\right) \qquad \epsilon \equiv \frac{I_{xx} - I_{yy}}{I_{zz}}$$

☐ Stochastic both from cosmological or astrophysical origin

$$[S_{GW}(f)]^{\frac{1}{2}} = (5.6 \times 10^{-22}) h_{100} (\Omega(f))^{\frac{1}{2}} \, (100\,\mathrm{Hz})^{\frac{3}{2}}\,\mathrm{Hz}^{-\frac{1}{2}}$$

GWs: more properties

- EM waves: at lowest order the radiation can be emitted by a dipole source (l=1). Monopolar radiation is forbidden as a result of charge conservation.

- GWs: the lowest allowed multipole is the quadrupole (l=2). The monopole is forbidden as a result of mass conservation. Similarly, dipole radiation is absent as a result of momentum conservation.

- GWs represents propagating "ripples in spacetime" or, more accurately, a propagating curvature perturbation. The perturbed curvature (Riemann tensor) is given by (in the TT gauge):

$$R^{\mathrm{TT}}_{j0k0} = -\frac{1}{2}\,\partial_t^2 h^{\mathrm{TT}}_{jk}\,, \qquad j,k = 1,2,3$$

GWs and curvature

- As we mentioned, GWs represent a fluctuating curvature field.
- Their effect on test particles is of tidal nature.
- Equation of geodetic deviation (in weak gravity):

$$\frac{d^2\xi^k}{dt^2} = -R^{k\ \ \mathrm{TT}}_{0j0}\,\xi^j$$

distance between geodesics (test particles)

- Newtonian limit: $R^{\mathrm{TT}}_{k0j0} \approx \dfrac{\partial^2\Phi}{\partial x^k \partial x^j}$

Newtonian grav. potential

GWs vs EM waves

- Similarities:

✓ Propagation with the speed of light.

✓ Amplitude decreases as ~ 1/r.

✓ Frequency redshift (Doppler, gravitational, cosmological).

- Differences:

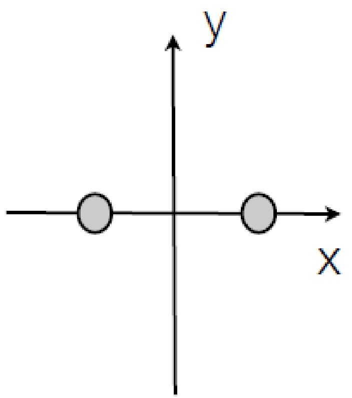
✓ GWs propagate through matter with little interaction. Hard to detect, but they carry uncontaminated information about their sources.

✓ Strong GWs are generated by bulk (coherent) motion. They require strong gravity/high velocities (compact objects like black holes and neutron star).

✓ EM waves originate from small-scale, incoherent motion of charged particles. They are subject to “environmental” contamination (interstellar absorption etc.).

Effect on test particles (I)

- We consider a pair of test particles on the cartesian axis Ox at distances $\pm x_0$ from the origin and we assume a GW traveling in the z-direction.



- Their distance will be given by the relation:

$$dl^2 = g_{\mu\nu}dx^\mu dx^\nu = ... = -g_{11}dx^2 =$$

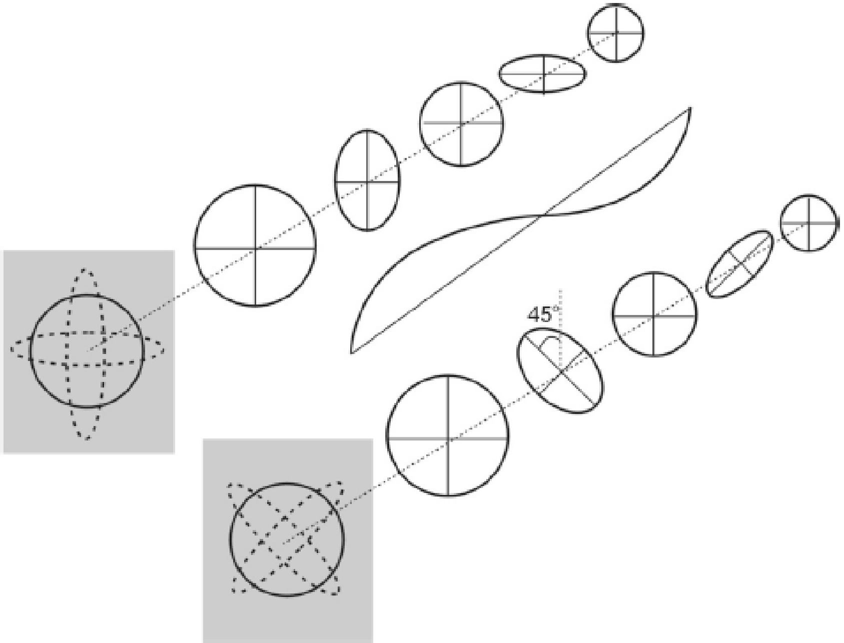
$$= (1 - h_{11})(2x_0)^2 = [1 - h_+ \cos(\omega t)]\,(2x_0)^2$$

$$dl \approx \left[1 - \frac{1}{2}h_+ \cos(\omega t)\right](2x_0)$$

Effect on test particles (II)

- Similarly for a pair of particles placed on the y-axis:
- Comment: the same result can be derived using the geodetic deviation equation.

$$dl \approx \left[1 + \frac{1}{2}h_+ \cos(\omega t)\right] (2y_0)$$



The quadrupole formula

- Einstein (1918) derived the quadrupole formula for gravitational radiation by solving the linearized field equations with a source term:

$$\Box h^{\mu\nu}(t,\vec{x}) = -\kappa T^{\mu\nu}(t,\vec{x}) \longrightarrow h^{\mu\nu} = -\frac{\kappa}{4\pi}\int_V d^3x' \frac{T^{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}$$

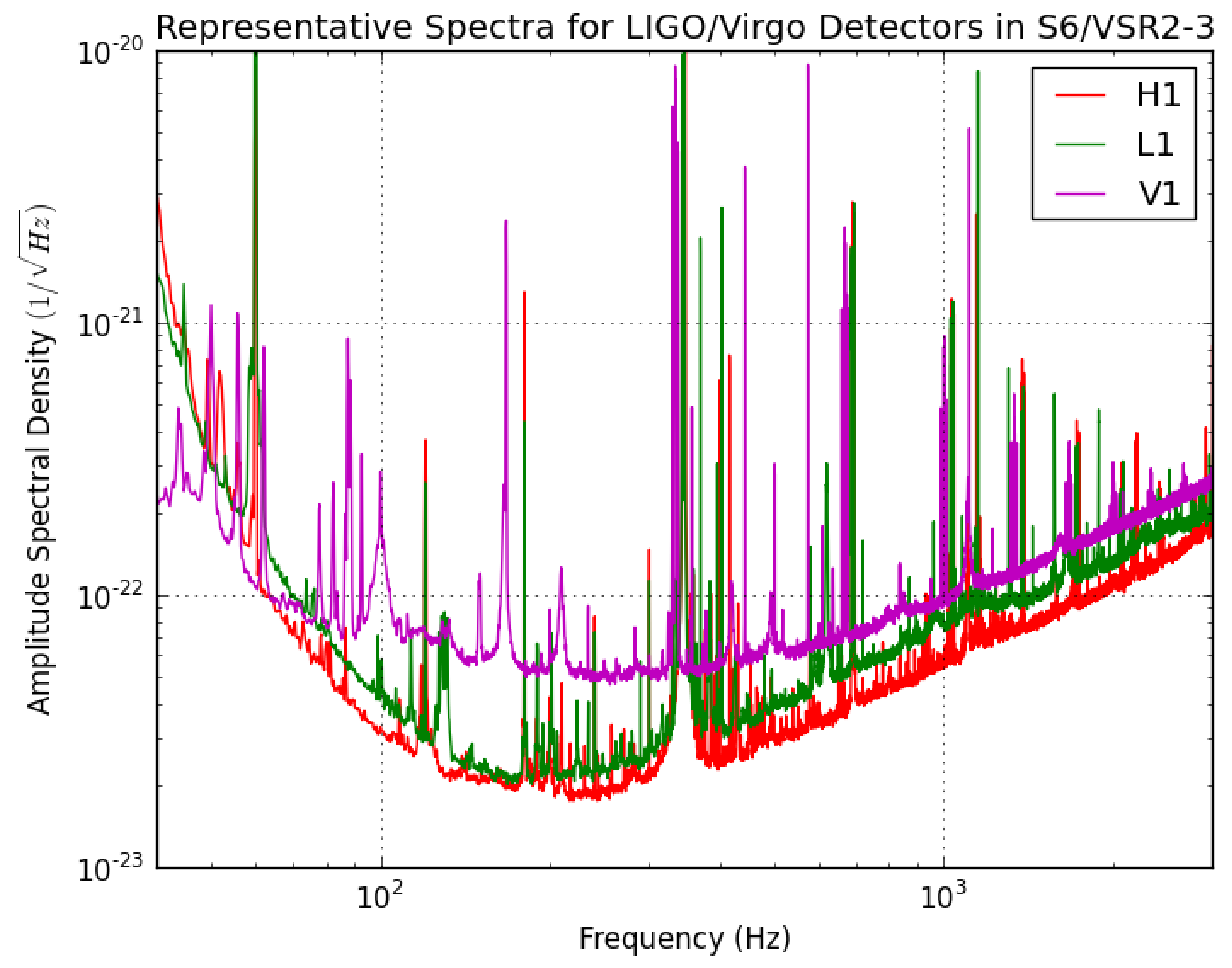
- This solution suggests that the wave amplitude is proportional to the second time derivative of the quadrupole moment of the source:

$$h^{\mu\nu} = \frac{2}{r}\frac{G}{c^4}\ddot{Q}^{\mu\nu}_{\mathrm{TT}}(t-r/c) \qquad Q^{\mu\nu}_{\mathrm{TT}} = \int d^3x\,\rho\left(x^\mu x^\nu - \frac{1}{3}\delta^{\mu\nu}r^2\right)$$

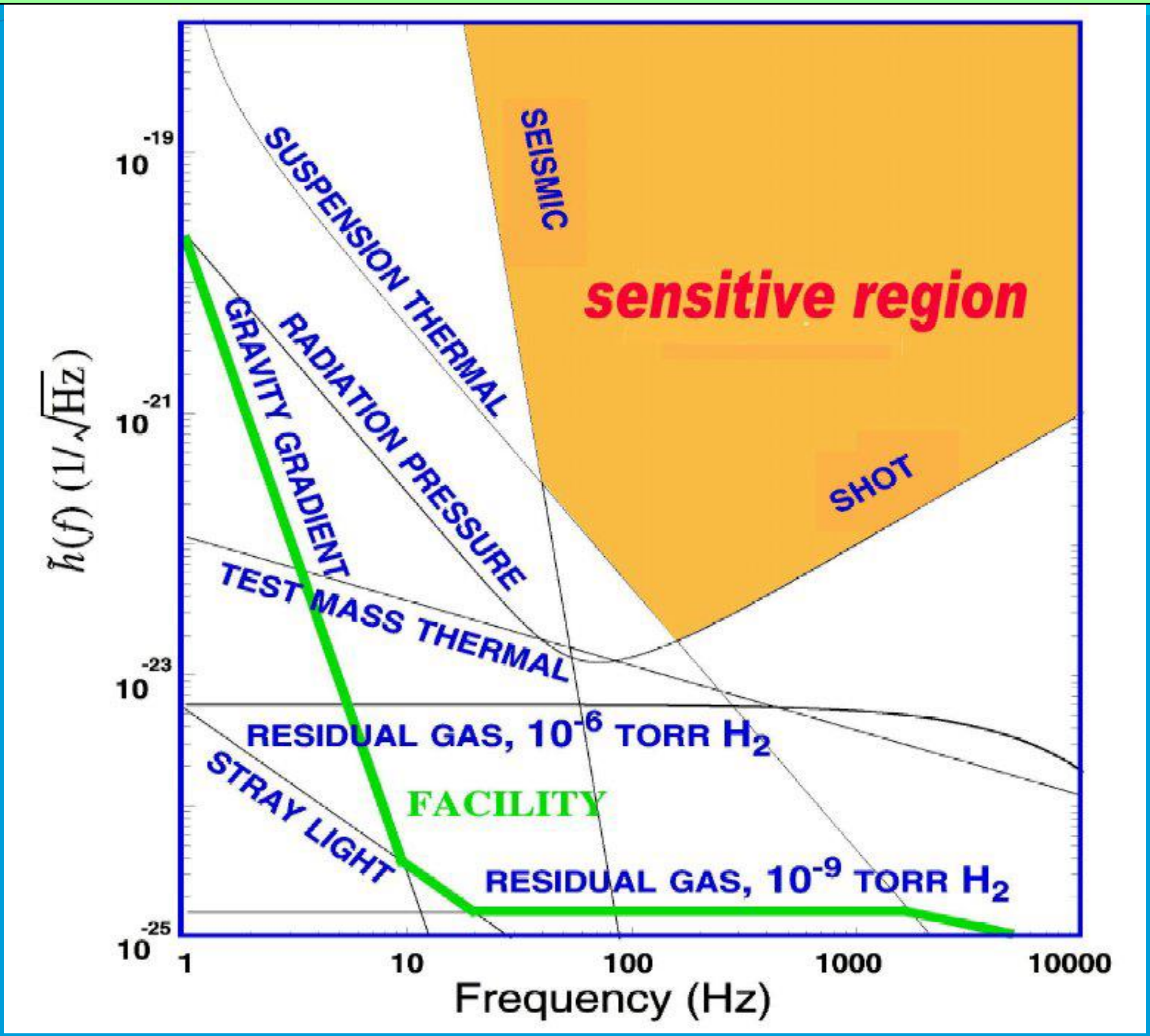
(quadrupole moment in the "TT gauge" and at the retarded time t-r/c)

- This result is quite accurate for all sources, as long as the wavelength is much longer than the source size R.

Sensitività degli osservatori Hanford (LIGO), Livingston (LIGO), Cascina (Virgo) nel 2012 circa

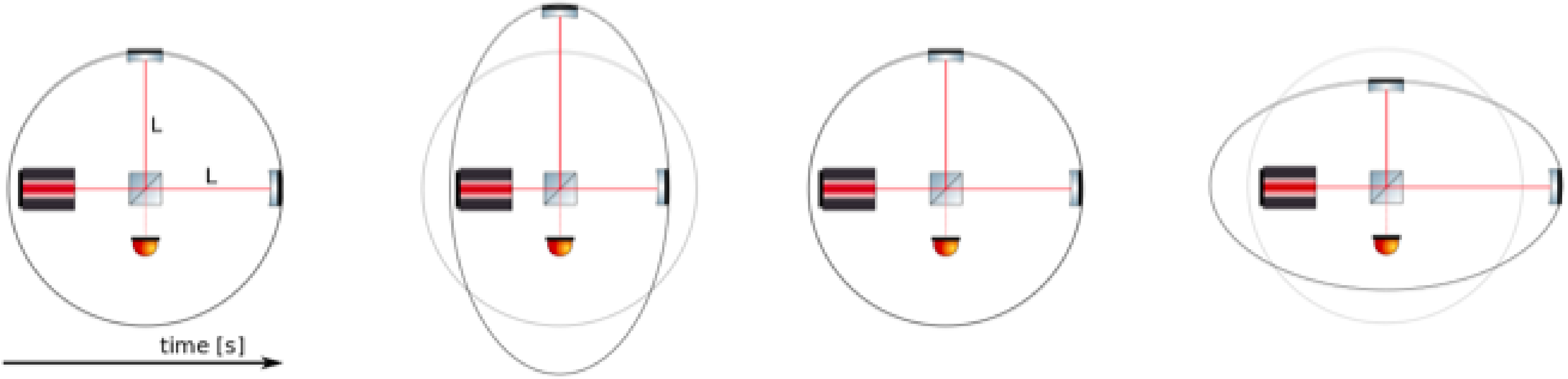


Noise structure in gravitational interferometers



Interferometers

Interferometer resopnse to h_+



$$\Delta\phi = 4\pi \frac{L h_+}{\lambda_{Laser}} \frac{\sin\left(\frac{\Omega_{GW} L}{c}\right)}{\frac{\Omega_{GW} L}{c}}$$

- Broad band
- Sensitivity increases with L
- Cutoff frequency:

$$\frac{\Omega_{GW} L}{c} \sim \frac{\pi}{2} \Rightarrow L < \frac{\lambda_{GW}}{4}$$

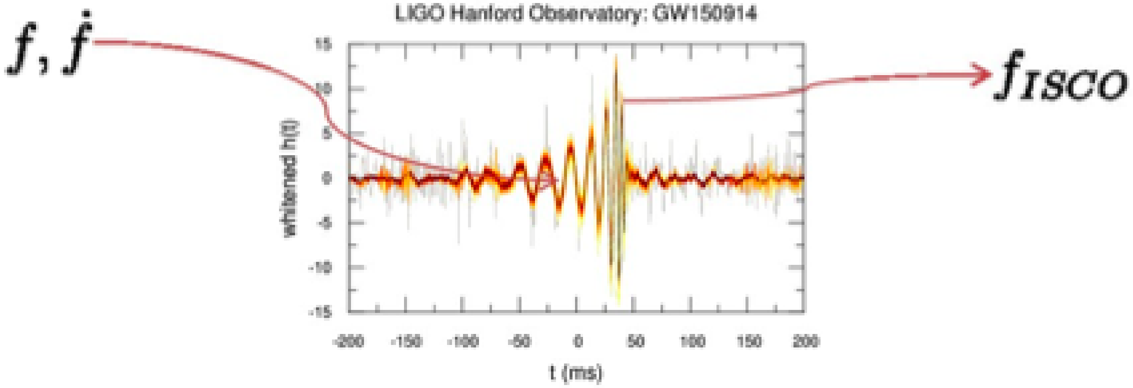
Anisotropic response and polarization sensitivity

$$h = F_{(+)} h_+ + F_{(\times)} h_\times$$

$$\begin{cases} h & \text{strain measured} \\ F_{(+,\times)} & \text{(sky position, polarization)} \end{cases}$$

GW150914 The binary parameters

Mass



Rough estimation: Leading order in the PN parameters and circular orbits

- Observed GW frequency f and time derivative $\dot{f}$ gives the «chirp mass» M_c

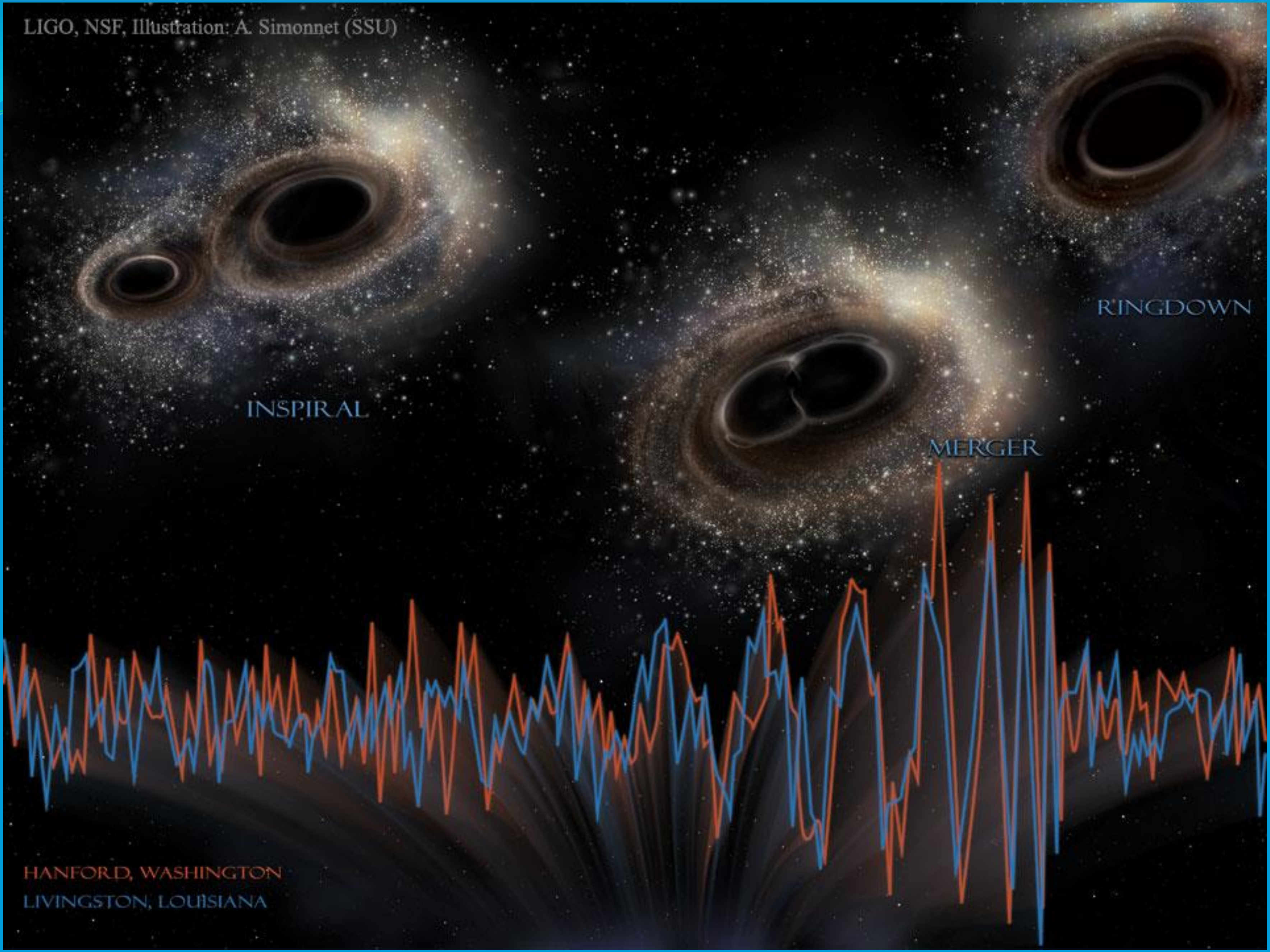
$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-3/8} f^{-11/3}\dot{f}\right]^{3/4} \approx 30\,\mathrm{M}_\odot$$

- Frequency of the Schwarzschild innermost stable circular orbit (ISCO) gives the total mass

$$f_{ISCO} \approx 4400\frac{\mathrm{M}_\odot}{M}[\mathrm{Hz}]$$

$$f_{ISCO} \approx 150\,\mathrm{Hz} \quad \Rightarrow \quad M_{Tot} = m_1 + m_2 \approx 70\,\mathrm{M}_\odot$$

$$m_1 \approx m_2 \approx 30\,\mathrm{M}_\odot$$


LIGO, NSF, Illustration: A. Simonnet (SSU)
INSPIRAL
MERGER
RINGDOWN
HANFORD, WASHINGTON
LIVINGSTON, LOUISIANA

High-Energy Events in the Universe

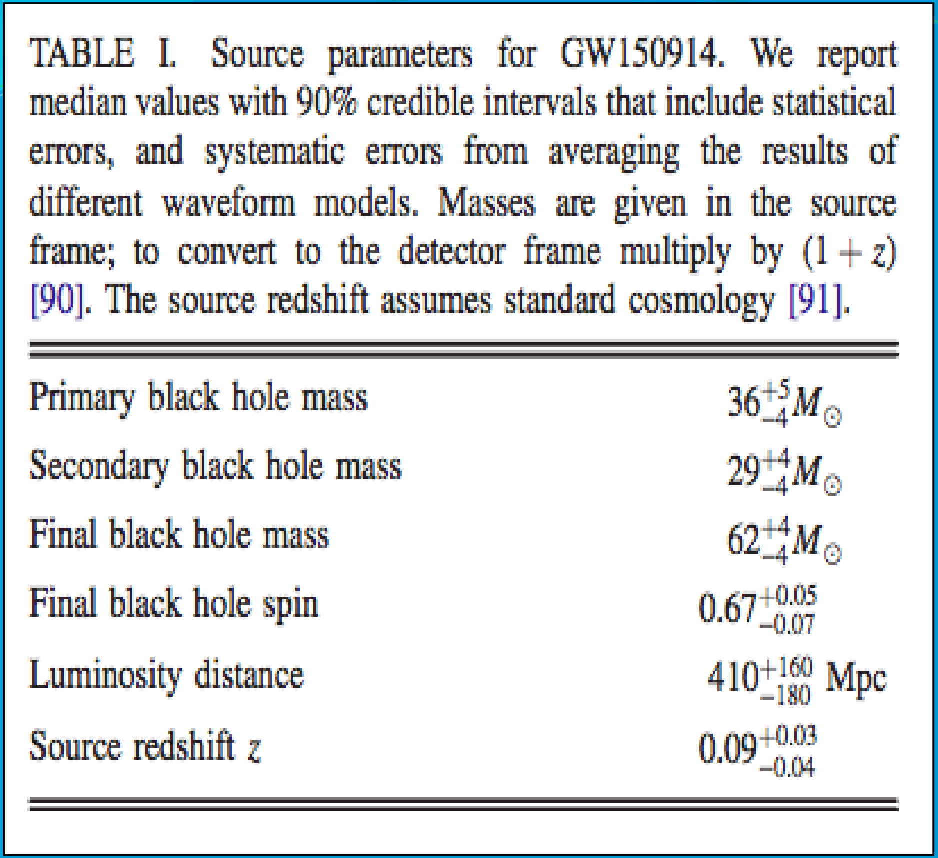
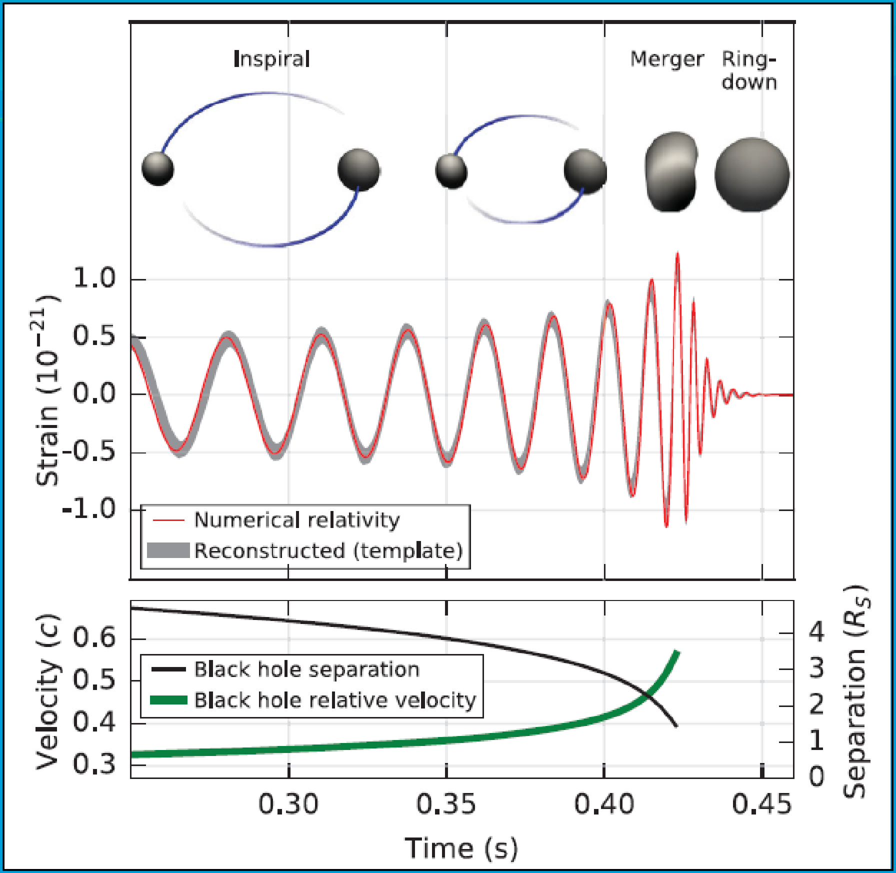


TABLE I. Source parameters for GW150914. We report median values with 90% credible intervals that include statistical errors, and systematic errors from averaging the results of different waveform models. Masses are given in the source frame; to convert to the detector frame multiply by $(1+z)$ [90]. The source redshift assumes standard cosmology [91].

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$



Maximum power transformed into gravitational waves is about 1000 times a Supernova energy release!

$$P_{\max} \approx 4\times10^{56}\ \ erg/s$$

Compare with Planck power

$$W_P = \frac{E_P}{t_P} = \sqrt{\frac{\hbar c^5}{2G}}\sqrt{\frac{c^5}{G\hbar}} = \frac{c^5}{G\sqrt{2}} = 2.5\times10^{59}\ \ erg\,s^{-1}$$

A Supernova

Total energy

$E \approx 0.05\, M_{sun} c^2 = 10^{53}\ erg$

90% of energy goes into neutrinos in 10 s

Avg Power

$P_{avg} \approx 10^{52}\ erg/s$

A Gamma-Ray Burst

Total energy

$E \approx 0.5\, M_{sun} c^2 = 10^{54}\ erg$

90% of energy goes into e.m. waves or gamma rays in 20 s

Avg Power

$P_{avg} \approx 5 \times 10^{52}\ erg/s$

GW150914

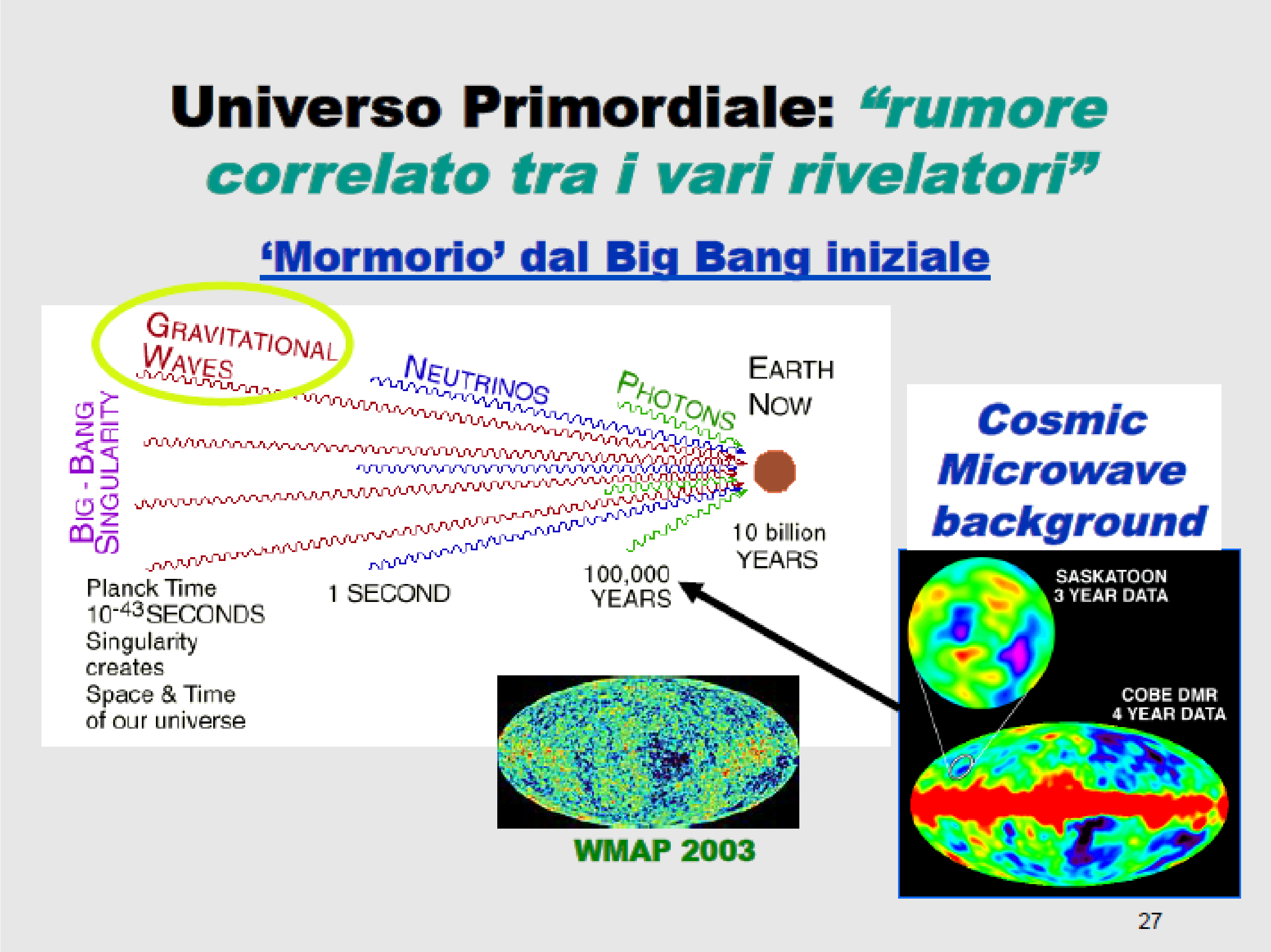
Total energy

$E \approx 3\, M_{sun} c^2 = 10^{55}\ erg$

All energy goes into gravitational waves in 0.5 s

Avg Power

$P_{avg} \approx 5 \times 10^{55}\ erg/s$

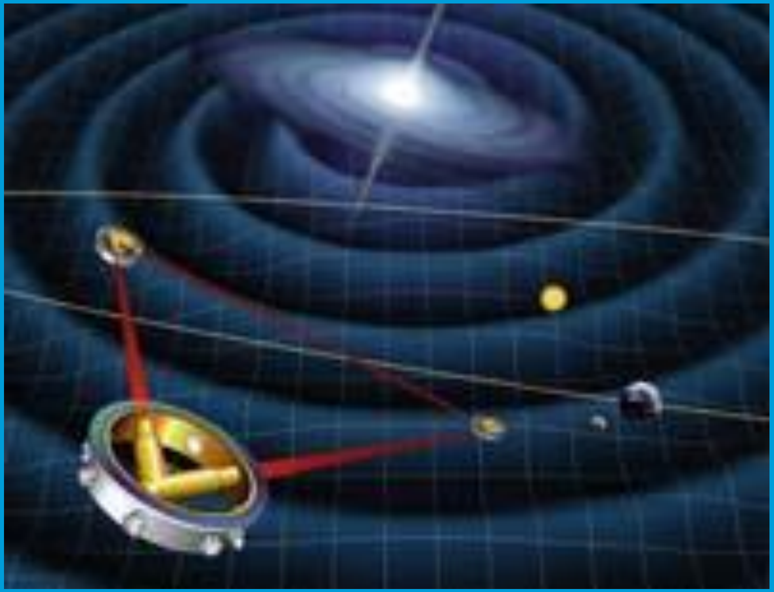


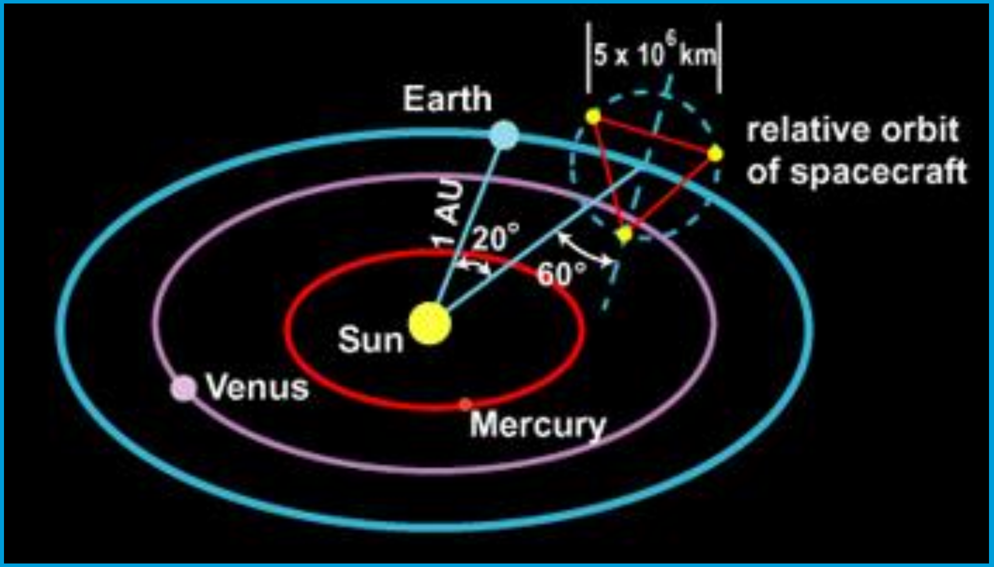
Universo Primordiale: *"rumore correlato tra i vari rivelatori"*

‘Mormorio’ dal Big Bang iniziale

Space-based GW detection

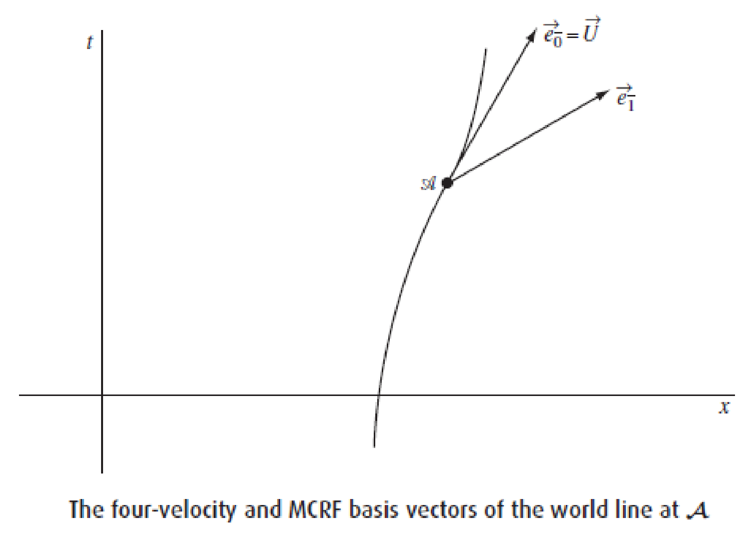
- LISA (*L*aser *I*nterferometer *S*pace *A*ntenna)





Four-Velocity and Four-Momentum

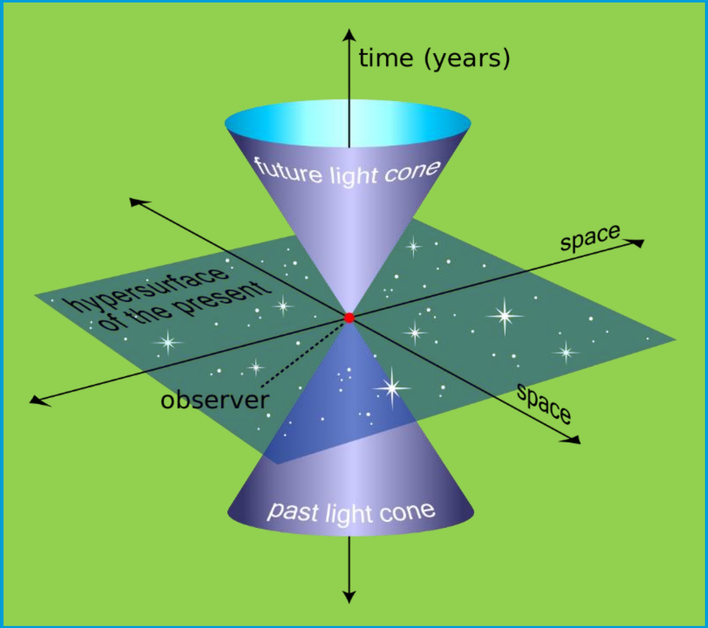
Four-velocity: a vector tangent to the world line of the particle, and stretching a unit time in the particle frame:



The four-velocity and MCRF basis vectors of the world line at $\mathcal{A}$

Let us consider a uniformly moving particle: in its rest frame the four-velocity points along the time axis and it is a unit time length

More in general: The four-velocity is the time basis vector of a particle in the MCRF



Let us define the four-momentum as

$$\vec{p} = m\vec{U}$$

And call its components in an inertial frame O

$$\vec{p} \rightarrow_O (E, p^1, p^2, p^3)$$

Let us now consider a particle in motion in an inertial system O and transform its four-velocity from the rest system Ó to the inertial system O (v along 1)

$$\vec{U} = \vec{e}_{\bar{0}} \qquad \vec{p} = m\vec{e}_{\bar{0}}$$

$$\vec{U} \rightarrow_{\bar{O}} (1,0,0,0)$$
$$\vec{p} \rightarrow_{\bar{O}} (m,0,0,0)$$

$$U^{\alpha} = \Lambda^{\alpha}_{\bar{\beta}} (\vec{e}_{\bar{0}})^{\bar{\beta}} = \Lambda^{\alpha}_{\bar{0}}$$
$$p^{\alpha} = m\Lambda^{\alpha}_{\bar{0}}$$

So that in the O system we have

$$p^{0} = mU^{0} = m\left(1 - v^{2}\right)^{-1/2} = m\gamma$$
$$p^{1} = mU^{1} = mv(1 - v^{2})^{-1/2} = mv\gamma$$
$$p^{2} = p^{3} = 0$$

Generaly Relativity Math

Metrics $\vec{A}\vec{B} = A^{\alpha} B^{\beta}\, \vec{e}_{\alpha}\vec{e}_{\beta} = A^{\alpha} B^{\beta} \eta_{\alpha\beta}$ (4-vectors) $\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$

$$ds^2 = dx^{\alpha} dx^{\beta} \eta_{\alpha\beta}$$

Curvilinear coordinates: Christoffel symbols (covariant derivative)

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \frac{\partial}{\partial x^{\beta}}\left(V^{\alpha}\vec{e}_{\alpha}\right) = \frac{\partial V^{\alpha}}{\partial x^{\beta}}\vec{e}_{\alpha} + V^{\alpha}\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = V^{\alpha}_{,\beta}\vec{e}_{\alpha} + V^{\alpha}\Gamma^{\mu}_{\alpha\beta}\vec{e}_{\mu} = (V^{\alpha}_{,\beta} + V^{\mu}\Gamma^{\alpha}_{\mu\beta})\,\vec{e}_{\alpha} = V^{\alpha}_{;\beta}\vec{e}_{\alpha}$$

$$V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + V^{\mu}\Gamma^{\alpha}_{\mu\beta}$$

$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_{\mu}\Gamma^{\mu}_{\alpha\beta}$$

$$g_{\alpha\beta;\mu} \equiv 0$$

Christoffel symbols and the metrics

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$$

Manifold: continuous space that locally looks (pseudo)Euclidean.
It is a set that can be continuously parametrized

Curvature can be introduced on the manifold

The surface of an infinite cylinder (or torus) is a manifold without curvature
The surface of a sphere is a manifold with curvature

We want only differentiable manifolds (a cone would not work, because of the apex)

Why introduce a metric over a differentiable manifold?
Because we need to express lengths and time intervals (and all physics)

A differentiable manifold on which a symmetric (0,2) tensor has been selected to act as the metrics is called a (pseudo)Riemannian manifold.

It is pseudo because the metrics is not positive-definite.

The metrics defines the curvature of the pseudo-Riemannian manifold.

Any symmetric matrix can be diagonalized preserving the number of positive and negative eigenvalues (the signature of the metrics).

Our capability to construct a local inertial frame:

$$g_{\mu\nu}(x) \rightarrow \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Local Flatness Theorem: at any point P on the manifold, a coordinate system can be found such that:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}(P) + O\left[(x^{\mu})^2\right]$$

(going to the local Lorent< frame). This is equivalent to:

$$g_{\mu\nu}(P) = \eta_{\mu\nu}$$

$$\frac{\partial}{\partial x^{\gamma}} g_{\mu\nu}(P) = 0$$

$$\frac{\partial^2}{\partial x^{\gamma} \partial x^{\mu}} g_{\mu\nu}(P) \neq 0$$

The curved space has a flat tangent space at any point
Straight lines in flat spacetimes are the world lines of free particles
So, free particles are moving on lines that are locally straight lines

$g_{\mu\nu,\gamma\mu}$ has 20 independent components

Lengths and Volumes :

$$l = \int \left| g_{\alpha\beta} dx^{\alpha} dx^{\beta} \right|^{1/2}$$

$$dx = (-g)^{1/2} dx' \qquad g \equiv \det(g_{\alpha\beta})$$

Going to the Lorentz local frame in P:

$$g_{\mu\nu}(P)=\eta_{\mu\nu} \qquad g_{\mu\nu,\gamma}(P)=0$$

$$\Gamma^{\gamma}_{\beta\mu}=\frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu}+g_{\alpha\mu,\beta}-g_{\beta\mu,\alpha}\right) \Rightarrow \Gamma^{\gamma}_{\beta\mu}(P)=0 \Rightarrow g_{\mu\nu,\gamma}(P)=g_{\mu\nu;\gamma}(P)=0$$

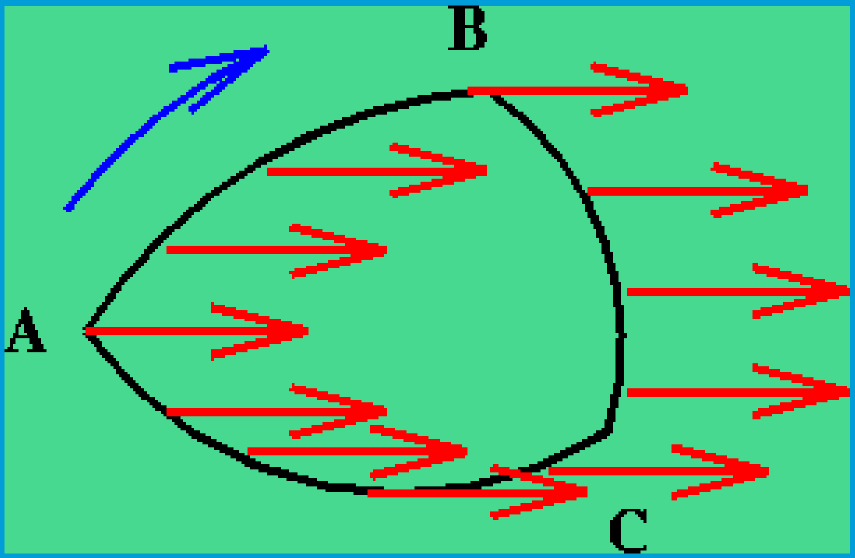
$$\Rightarrow g_{\mu\nu;\gamma}=0$$

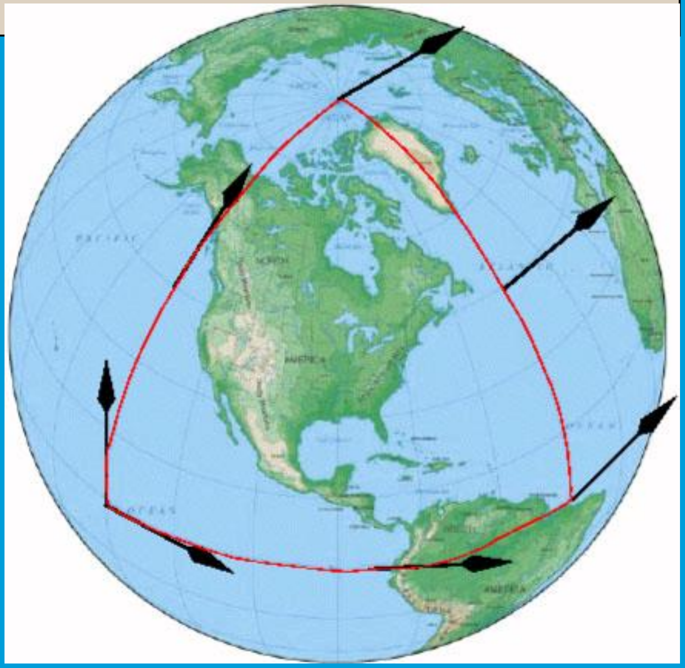
Now considering curvature :

- Extrinsic (just due to immersion in higher dimensional space). Euclid theorems holds and the space can be deformed (without tearing it or crumpling it) into flat space. Example: the cylinder.

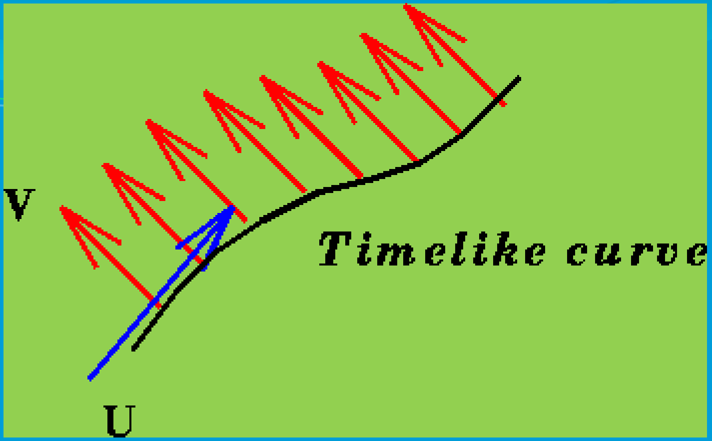
- Intrinsic, like the surface of the sphere: parallel lines, when continued, do not remain parallel.

Paraller transport along a closed path: difference between a flat space (Euclidean plane) and a curved space (surface of the sphere).





Parallel transporting 4-vector V :



In a locally Lorentz frame :

$$\frac{dV^{\alpha}}{d\lambda}=0 \rightarrow \frac{dV^{\alpha}}{dx^{\beta}}\frac{dx^{\beta}}{d\lambda}=V^{\alpha}_{,\beta}U^{\beta}=0$$

The equation for parallel transport is then :

$$U^{\beta}V^{\alpha}_{;\beta}=0 \Leftrightarrow \nabla_{U}\vec{V}=0$$

Geodesics : a curve that parallel-transports its own tangent vector :

$$\nabla_{U}\vec{U}=0$$

$$U^{\beta}U^{\alpha}_{;\beta}=U^{\beta}U^{\alpha}_{,\beta}+U^{\beta}U^{\mu}\Gamma^{\alpha}_{\mu\beta}=0$$

If λ is a curve parameter this becomes :

$$U^{\alpha}=\frac{dx^{\alpha}}{d\lambda} \qquad U^{\beta}\frac{\partial}{\partial x^{\beta}}=\frac{dx^{\beta}}{d\lambda}\frac{d}{dx^{\beta}}=\frac{d}{d\lambda}$$

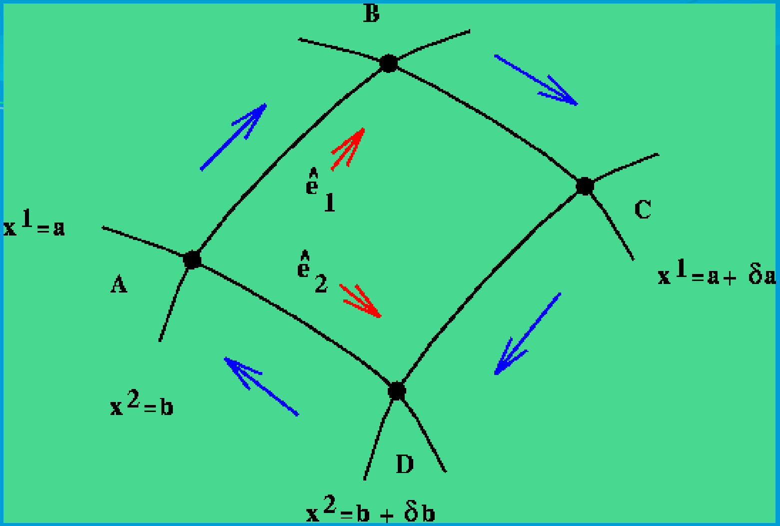
$$\Rightarrow \frac{dU^{\alpha}}{d\lambda}+\Gamma^{\alpha}_{\mu\beta}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\mu}}{d\lambda}=0$$

$$\frac{d}{d\lambda}\frac{dx^{\alpha}}{d\lambda}+\Gamma^{\alpha}_{\mu\beta}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\mu}}{d\lambda}=0$$

Geodesic Equation. It is a curve of extremal length

Curvature Tensor

Curvature: the effect of parallel transport around a closed loop



$$\delta V^{\alpha} = \delta a\,\delta b\,V^{\mu}\left[\Gamma^{\alpha}_{\mu\sigma,\lambda} - \Gamma^{\alpha}_{\mu\lambda,\sigma} + \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\nu}_{\mu\sigma} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\nu}_{\mu\lambda}\right]$$
$$+\delta a\,\vec{e}_{\sigma} + \delta b\,\vec{e}_{\lambda} - \delta a\,\vec{e}_{\sigma} - \delta b\,\vec{e}_{\lambda}$$

$$\delta V^{\alpha} = \delta a\,\delta b\,V^{\mu}R^{\alpha}_{\mu\lambda\sigma}$$

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

Riemann Curvature Tensor

What is the form of the Riemann Curvature Tensor in the local Lorentz frame P ?

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right) \Rightarrow \Gamma^{\gamma}_{\beta\mu}(P) = 0 \Rightarrow g_{\mu\nu,\gamma}(P) = g_{\mu\nu;\gamma}(P) = 0$$

But the derivative of Γ is not zero

$$\Gamma^{\alpha}_{\mu\nu,\sigma}(P) = \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma}\right)$$

$$R^{\alpha}_{\beta\mu\nu}(P) = \frac{1}{2}g^{\alpha\sigma}\left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}\right)$$

The following identities can be derived (which are true in P, but being tensor equations are valid everywhere)

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

Given these equations, R has 20 independent components (in 4 dimensions)

Flat manifold ⟺ $R^{\alpha}_{\beta\mu\nu} = 0$

Commutation of covariant derivatives and curvature

$$\nabla_\beta A^\alpha = A^\alpha_{;\beta} = A^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} A^\mu$$

Now, let us do

$$\nabla_\alpha \nabla_\beta V^\mu = \nabla_\alpha V^\mu_{;\beta} = \left(V^\mu_{;\beta}\right)_{,\alpha} + V^\sigma_{;\beta}\,\Gamma^\mu_{\sigma\alpha} - V^\mu_{;\sigma}\,\Gamma^\sigma_{\beta\alpha}$$

In a locally inertial frame P, only the first term contribute, since the others will never contain derivates of Γ :

$$\nabla_\alpha \nabla_\beta V^\mu(P) = \left(V^\mu_{;\beta}\right)_{,\alpha} = V^\mu_{,\beta\alpha} + \Gamma^\mu_{\nu\beta,\alpha} V^\nu$$

And also

$$\nabla_\beta \nabla_\alpha V^\mu(P) = V^\mu_{,\alpha\beta} + \Gamma^\mu_{\nu\alpha,\beta} V^\nu$$

The commutator

$$\left[\nabla_\alpha, \nabla_\beta\right] V^\mu(P) = V^\nu \left(\Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta}\right)$$

In P the Riemann tensor

$$R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} \to_P \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu}$$

so that, in the locally inertial frame P :

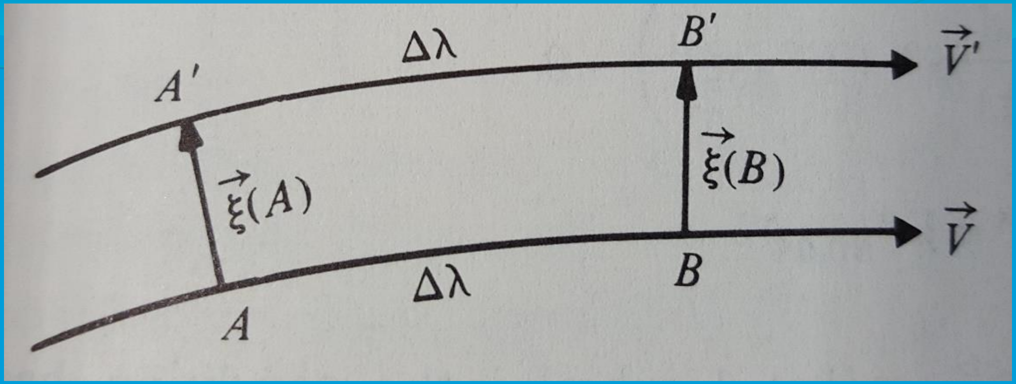
$$\left[\nabla_\alpha, \nabla_\beta\right] V^\mu(P) = R^\mu_{\nu\alpha\beta} V^\nu$$

This is a valid tensor equation → valid in any coordinate system.

Geodesic Deviation

Parallel lines, when extended, do not remain parallel.

Now, suppose they start parallel at A and A’ :



Equation of the geodesics at A (assumed to be locally inertial), assuming that x^0 points along the geodesics ($V^\alpha = \delta^\alpha_0$)

$$\frac{d}{d\lambda}\frac{dx^{\alpha}}{d\lambda}_{(A)} = 0$$

Equation of the other geodesics at A’, under the same assumptions (except local inertiality):

$$\frac{d}{d\lambda}\frac{dx^{\alpha}}{d\lambda}_{(A')} + \Gamma^{\alpha}_{00}(A') = 0$$

$$\Gamma^{\alpha}_{00}(A') \cong \Gamma^{\alpha}_{00,\beta}\,\xi^{\beta}$$

$$\frac{d}{d\lambda}\frac{dx^{\alpha}}{d\lambda}_{(A')} + \Gamma^{\alpha}_{00,\beta}\xi^{\beta} = 0$$

$$\frac{d^{2}\xi^{\alpha}}{d\lambda^{2}} = \frac{d^{2}x^{\alpha}}{d\lambda^{2}}_{(A')} - \frac{d^{2}x^{\alpha}}{d\lambda^{2}}_{(A)} = -\Gamma^{\alpha}_{00,\beta}\xi^{\beta}$$

Now it is necessary to build the full second covariant derivative

$$\nabla_V \nabla_V \xi^\alpha = \frac{d}{d\lambda}\left(\nabla_V \xi^\alpha\right) + \Gamma^\alpha_{\beta 0}\left(\nabla_V \xi^\beta\right)$$

Now using the fact that in A: Γ=0

$$\nabla_V \nabla_V \xi^\alpha = \frac{d}{d\lambda}\left(\frac{d}{d\lambda}\xi^\alpha + \Gamma^\alpha_{\beta 0}\xi^\beta\right) + 0 = \frac{d^2}{d\lambda^2}\xi^\alpha + \Gamma$$

Bianchi Identity, Ricci Tensor, Einstein Tensor

Starting from the Riemann Tensor:

$$R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}\right)$$

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\mu\nu} = \frac{1}{2} g_{\alpha\lambda} g^{\lambda\sigma}\left(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}\right) = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu})$$

And derivating:

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda})$$

Using the symmetry properties of g and the commutation of partial derivatives

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0$$

A valid tensor equation which is valid everywhere

$$R_{\alpha\beta\mu\nu\,;\lambda} + R_{\alpha\beta\lambda\mu\,;\nu} + R_{\alpha\beta\nu\lambda\,;\mu} = 0$$

Bianchi Identity

Ricci Tensor

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta} = R_{\beta\alpha}$$

It is the only independent contraction of the Riemann Curvature Tensor

Ricci Scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}_{\mu\alpha\nu} = g^{\mu\nu}g^{\alpha\beta}R_{\alpha\mu\beta\nu}$$

Einstein Tensor

Starting with the Bianchi Identity

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$$

$$g^{\alpha\mu}\left(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}\right) = 0$$

this contraction yields

$$R^{\mu}_{\beta\mu\nu;\lambda} + R^{\mu}_{\beta\lambda\mu;\nu} + R^{\mu}_{\beta\nu\lambda;\mu} = 0$$

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu} = 0$$

yet another contraction!

$$g^{\beta\nu}\left(R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu}\right) = R_{;\lambda} - R^{\nu}_{\lambda;\nu} + R^{\mu\nu}_{\nu\lambda;\mu} = R_{;\lambda} - R^{\mu}_{\lambda;\mu} - R^{\mu}_{\lambda;\mu} = 0$$

$$2R^{\mu}_{\lambda;\mu} - \delta^{\mu}_{\lambda}R_{;\mu} = 0$$

$$R^{\mu}_{\lambda;\mu} - \frac{1}{2}\delta^{\mu}_{\lambda}R_{;\mu} = 0$$

$$g^{\alpha\lambda}\left(R^{\mu}_{\lambda} - \frac{1}{2}\delta^{\mu}_{\lambda}R\right)_{;\mu} = 0$$

$$\left(R^{\mu\alpha} - \frac{1}{2}g^{\alpha\mu}R\right)_{;\mu} = 0$$

$$\left(R^{\alpha\mu} - \frac{1}{2}g^{\alpha\mu}R\right)_{;\mu} = 0$$

$$G^{\alpha\beta} = \left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)$$

$$G^{\alpha\beta}_{;\beta} = 0$$

Einstein Equations for Weak Fields

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Flat spacetime}} + \underbrace{h_{\mu\nu}}_{\text{curvature}}$$

$$\left|h_{\mu\nu}\right| << 1$$

Everywhere in spacetime

A coordinate system in which this happens is called a Nearly Lorentz Coordinate System

If one such system exists, then there are many others, linked by

1) Background Lorentz : $x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} x^{\beta}$ where the matrix Λ is a Special Relativity transformation. Under this transformation, one has $g_{\bar{\mu}\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} + h_{\bar{\mu}\bar{\nu}}$ with h transforming as a tensor. This leads to the convenient fiction of thinking of a flat spacetime with a tensor defined on it.

2) Gauge transformations : small change of coordinate generated by a vector $x^{\alpha'} = x^{\alpha} + \xi^{\alpha}(x^{\beta})$ so that $g_{\alpha'\beta'} = \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$ (the effect of coordinate changes is to redefine h).